Arithmetic and Geometric SEQUENCES

MODULAR SYSTEM

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Arithmetic and Geometric SEQUENCES

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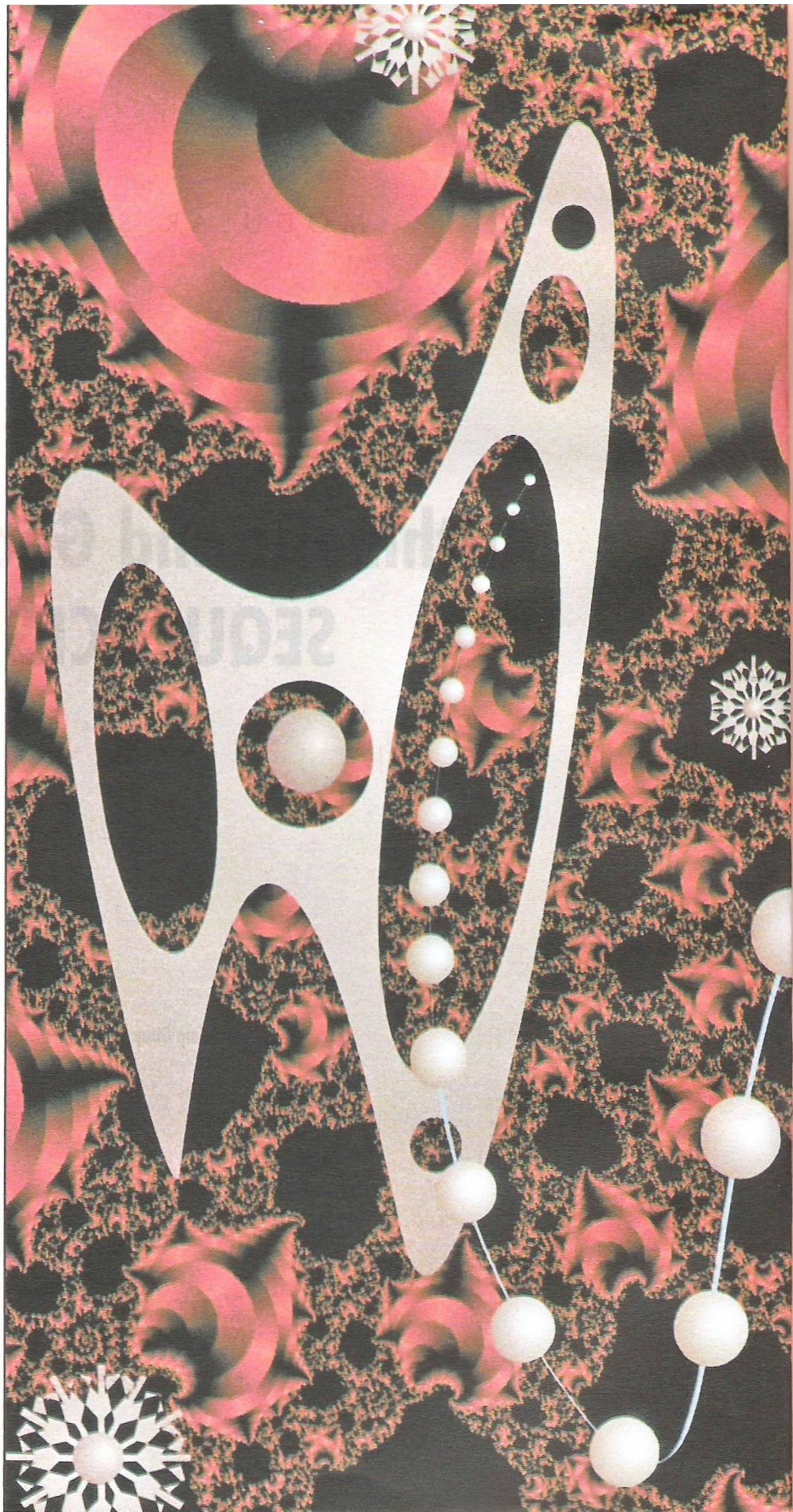


PREFACE

038

To the Teacher

This book is about arithmetic and geometric sequences, and their applications. Many people apply the principles of number sequences in their daily lives without recognizing them. This book introduces these principles and shows how they solve concrete problems. Sequences have important applications in communication systems, global positioning systems, software testing, cryptography, computer simulation, and stream ciphers. For many students these tools and the algorithms used to develop them will be as fundamental in their professional work as the tools of mathematical analysis. For this reason, this book goes beyond a purely analytical approach to sequences, and draws on techniques and examples from applied math and mathematical modeling.

The language of the book is student-friendly more than purely mathematical. It tries to explain the topic as a teacher would explain it in the classroom, so some exercises prompt the student to think for him or herself. Since the book focuses on developing algorithms and modeling applications, the examples do not require complex calculations.

The book is divided into three sections. The first section, real number sequences, deals with general number sequences with a specific pattern, and forms the basis of the book. In the second and third sections we study two of the most frequent types of sequence, arithmetic and geometric, consecutively.

Each section is followed by plenty of exercises. More difficult problems are denoted by a single or double star, where the former means problems for upper-intermediate level students, and the latter means problems at advanced level. Most of the problems reflect skills or problem-solving techniques encountered in the section. Every exercise set also contains problems whose solution method is not covered in an example. In these problems students may be required to work a little beyond the material discussed in the text, or to use the concepts in ways not illustrated in the examples. All of these problems can be solved using skills the student should already have mastered.

Following each section we discuss an activity or project related to the material covered. The topics are the Fibonacci sequence, polygonal numbers, magic squares, the Sierpinski pyramid, and the Koch snowflake. These sections can be used as term projects to increase the students' understanding of the topic.

The book follows a linear approach, with material in the latter sections building on concepts and math covered previously in the text. For this reason, there are several self-test 'Check Yourself' sections that check students' understanding of the material at key points. 'Check Yourself' sections include a rapid answer key that allows students to measure their own performance and understanding. Successful completion of each self-test section allows students to advance to the next topic.

The book ends with review materials, beginning with a brief summary of the chapter highlights. Following these highlights is a concept check test that asks the student to summarize the main ideas covered in the book. Following the concept check, review tests cover material from the entire book.

Acknowledgements

Many friends and colleagues were of great help in writing this textbook. A number of people need to be recognized and thanked for their contributions, including Mustafa Kırıkçı at Zambak Publications, and Serdar Çam for his typesetting and design.

Cem Giray

To the Student

This book is designed so that you can use it effectively. Each section has its own special color that you can see at the bottom of the page.

Section 1 — Real Number Sequences

Section 2 — Arithmetic Sequences

Section 3 — Geometric Sequences

Different pieces of information in this book are useful in different ways. Look at the types of information, and how they appear in the book:

Note
A function is a correspondence bet
A exactly one element of set B

Notes help you focus on important details. When you see a note, read it twice! Make sure you understand it.

Definition boxes give a formal description of a new concept. Notation boxes explain the mathematical way of expressing concepts. Theorem boxes include propositions that can be proved. The information in these boxes is very important for further understanding and for solving examples.

Theorem
The infinite

Notation
We denote a sequ
of a sequence as

Definition
sequence
A function which

EXAMPLE 36 Given the arithmetic sequence w

Solution $S_3 = a_1 + a_2 + a_3 = 4 + 7 + 1($

Examples include problems related to the topic and their solution, with explanations. The examples are numbered, so you can find them easily in the book.

Check Yourself sections help you check your understanding of what you have just studied. Solve them alone and then check your answers against the answer key provided. If your answers are correct, you can move on to the next section.

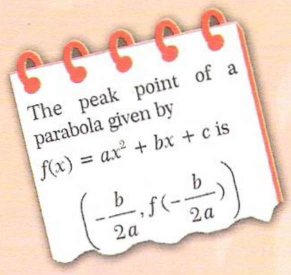
Check Yourself 1

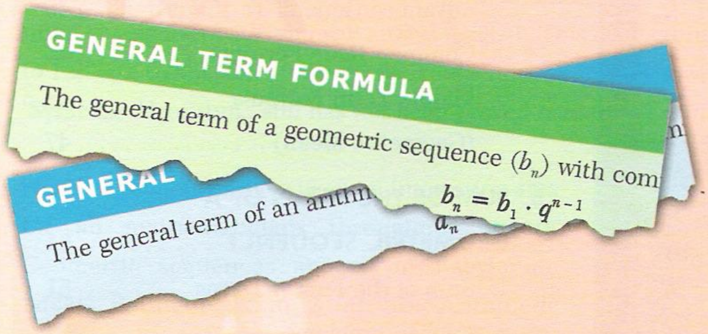
1. Write the first five terms of the sequence wh
2. Find a suitable general term a_n for the seque
3. Given the sequence with general term $b_n = 2$

Answers

1. –1, 1, –1, 1, –1 2. $2n$ 3. 13, undefined, 89

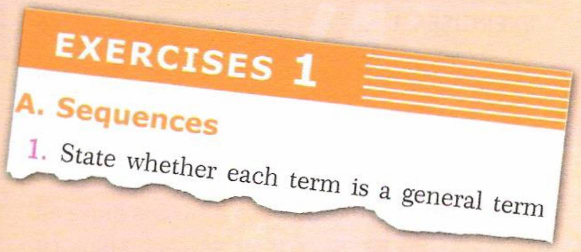
A small notebook in the left margin of a page reminds you of material that is related to the topic you are studying. Notebook text helps you to remember the math you need to understand the material. It might help you to see your mistakes, too! Notebooks are the same color as the section you are studying.

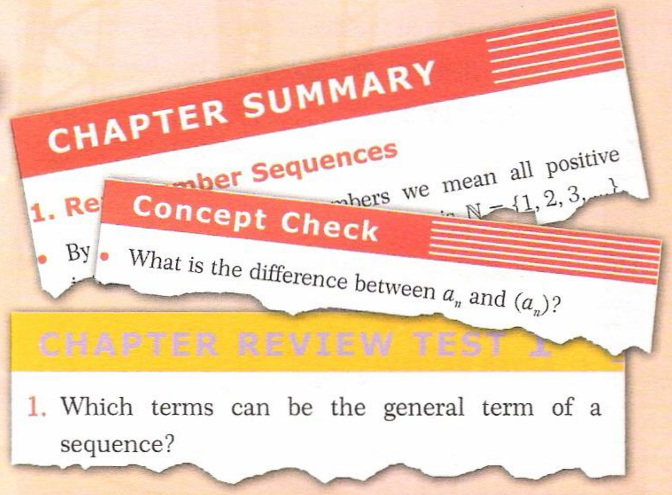




Special windows highlight important new information. Windows may contain formulas, properties, or solution procedures, etc. They are the same color as the section.

Exercises at the end of each section cover the material in the whole section. You should be able to solve all the problems without any special symbol. ✪ next to a question means the question is a bit harder. ✪✪ next to a question means the question is for students who are looking for a challenge! The answers to the exercises are at the back of the book.





The Chapter Summary summarizes all the important material that has been covered in the chapter. The Concept Check section contains oral questions. In order to answer them you don't need paper or pen. If you answer Concept Check questions correctly, it means you know that topic!

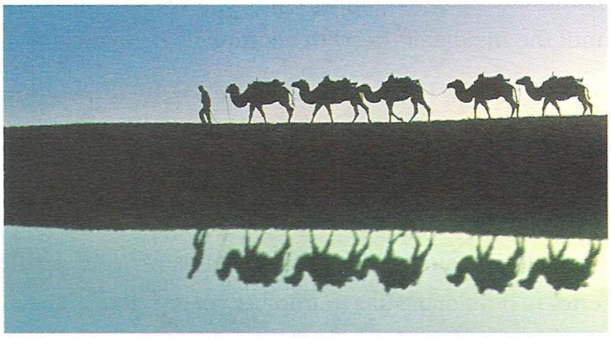
The answers to Concept Check questions are in the material you studied. Go back over the material if you are not sure about an answer to a Concept Check question.

Finally, chapter review tests are in increasing order of difficulty and contain multiple choice questions. The answer key for these tests is at the back of the book.

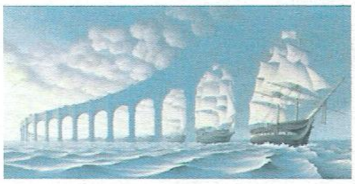
CONTENTS

INTRODUCTION...

An interesting unsolved problem in mathematics concerns the 'hailstone sequence', which is defined as follows: Start with any positive integer. If that number is odd, then multiply it by three and add one. If it is even, divide it by two. Then repeat. For example, starting with the number 10 we get the hailstone sequence 10, 5, 16, 8, 4, 2, 1, 4, 2, 1... . Some mathematicians have *conjectured* (guessed) that no matter what number you start with, you will always reach 1. This conjecture has been found true for all starting values up to 1,200,000,000,000. However, the conjecture, which is known as the 'Collatz Problem', '3n+1 Problem', or 'Syracuse Algorithm', still has not been proved true for all numbers.



Number sequences have been an interesting area for all mathematicians throughout history. Geometric sequences appear on Babylonian tablets dating back to 2100 BC. Arithmetic sequences were first found in the Ahmes Papyrus which is dated at 1550 BC. The reason behind the names 'arithmetic' and 'geometric' is that each term in a geometric (or arithmetic) sequence is the geometric (or arithmetic) mean of its successor and predecessor. If we think of a rectangle with side lengths x and y, then the geometric mean $\sqrt{xy}$ is the side length of a square that has the same area as this rectangle. Finding the dimensions of a square with the same area as a given rectangle was considered in those days as a very geometric problem. Although the arithmetic mean $(x + y)/2$ can also be interpreted geometrically (it is the length of the sides of a square having the same perimeter as the rectangle), lengths were viewed more as arithmetic, because it is easier to handle lengths by addition and subtraction, without having to think about two-dimensional concepts such as area. Although both problems involve arithmetic and can be interpreted geometrically, in ancient times one was viewed as much more geometric than the other, therefore the names.



Zeno (490-425 B.C.) was a mathematician whose paradoxes about motion puzzled mathematicians for centuries. They involved the sum of an infinite number of positive terms to a finite number. Zeno wasn't the only ancient mathematician to work on sequences. Several of the ancient Greek mathematicians used sequences to measure areas and volumes of shapes and regions. By using his reasoning technique called the 'method', Archimedes (287-212 B.C.) constructed several examples and tried to explain how infinite sums could have finite results. Among his many results was that the area under a parabolic arc is always two-thirds the base times the height.

The next major contributor to this area of mathematics was Fibonacci (1170-1240). He discovered a sequence of integers in which each number is equal to the sum of the preceding two numbers (1, 1, 2, 3, 5, 8, ...), and introduced it as a model of the breeding population of rabbits. This sequence has many remarkable properties and continues to find applications in many areas of modern mathematics and science. During this same period, Chinese astronomers developed numerical techniques to analyze their observation data and used the idea of finite differences to help analyze trends in their data.

Oresme (1325-1382) studied rates of change, such as velocity and acceleration, using sequences. Two hundred years later, Stevin (1548-1620) understood the physical and mathematical conceptions of acceleration due to gravity using series and sequences. During that time Galileo (1564-1642) applied mathematics to the sciences, especially astronomy. Based on his study of Archimedes, Galileo improved our understanding of hydrostatics. He developed equations for free-fall motion under gravity and the motion of the planets. Up until the middle of the 17th century, mathematicians developed and analyzed series of numbers.

Newton (1642-1727) and Leibnitz (1646-1716) developed several series representations for functions. Maclaurin (1698-1746), Euler (1707-1783), and Fourier (1768-1830) often used infinite series to develop new methods in mathematics. Sequences and series have become standard tools for approximating functions and calculating results in numerical computing.

The self-educated Indian mathematician Srinivasa Ramanujan (1887-1920) used sequences and power series to develop results in number theory. Ramanujan's work was theoretical and produced many important results used by mathematicians in the 20th century.

ARITHMETIC AND GEOMETRIC SEQUENCES

1 REAL NUMBER SEQUENCES

Real number sequences are strings of numbers. They play an important role in our everyday lives. For example, the following sequence:

$$20, 20.5, 21, 22, 23.4, 23.6, \ldots$$

gives the temperature measured in a city at midday for five consecutive days. It looks like the temperature is rising, but it is not possible to exactly predict the future temperature.

The sequence:

$$64, 32, 16, 8, \ldots$$

is the number of teams which play in each round of a tournament so that at the end of each game one team is eliminated and the other qualifies for the next round. Now we can easily predict the next numbers: 4, 2, and 1. Since there will be one champion, the sequence will end at 1, that is, the sequence has a finite number of terms. Sequences may be finite in number or infinite.

Look at the following sequence:

$$1000, 1100, 1210, \ldots$$

This is the total money owned by an investor at the end of each successive year. The capital increases by 10% every year. You can predict the next number in the sequence to be 1331. Each successive term here is 110% of, or 1.1 times, the previous term.

Can you recognize the pattern?

Real number sequences may follow an easily recognizable pattern or they may not. Recently a great deal of mathematical work has concentrated on deciding whether certain number sequences follow a pattern (that is, we can predict consecutive terms) or whether they are random (that is, we cannot predict consecutive terms). This work forms the basis of chaos theory, speech recognition, weather prediction and financial management, which are just a few examples of an almost endless list. In this book we will consider real number sequences which follow a pattern.

A. SEQUENCES

1. Definition

By the set of natural numbers we mean all positive integers and denote this set by $\mathbb{N}$.

That is, $\mathbb{N} = \{1, 2, 3, ...\}$.

If someone asked you to list the squares of all the natural numbers, you might begin by writing

$$1, 4, 9, 16, 25, 36, ...$$

But you would soon realize that it is actually impossible to list all these numbers since there are an infinite number of them. However, we can represent this collection of numbers in several different ways.

A function is a relation between two sets A and B that assigns to each element of set A exactly one element of set B.

For example, we can also express the above list of numbers by writing

$$f(1), f(2), f(3), f(4), f(5), f(6), ..., f(n), ...$$

where $f(n) = n^2$. Here $f(1)$ is the first term, $f(2)$ is the second term, and so on. $f(n) = n^2$ is a **function** of n, defined in the set of natural numbers.

Definition

sequence

A function which is defined in the set of natural numbers is called a **sequence.**

However, we do not usually use functional notation to describe sequences. Instead, we denote the first term by a_1, the second term by a_2, and so on. So for the above list

$$a_1 = 1,\ a_2 = 4,\ a_3 = 9,\ a_4 = 16,\ a_5 = 25,\ a_6 = 36,\ ...,\ a_n = n^2,\ ...$$

Here, a_1 is the first term,

a_2 is the second term,

a_3 is the third term,

$\vdots$

a_n is the nth term, or the **general term**.

Since this is just a matter of notation, we can use another letter instead of the letter a. For example, we can also use b_n, c_n, d_n, etc. as the name for the general term of a sequence.

Notation

We denote a sequence by (a_n), where a_n is written inside brackets. We write the general term of a sequence as a_n, where a_n is written without brackets. For the above example, if we write the general term, we write $a_n = n^2$.

If we want to list the terms, we write $(a_n) = (1, 4, 9, 16, ..., n^2, ...)$.

Sometimes we can also use a shorthand way to write a sequence:

$(a_n) = (n^2 + 4n + 1)$ means the sequence (a_n) with general term $a_n = n^2 + 4n + 1$.

Note
An expression like $a_{2.6}$ is nonsense since we cannot talk about the 2.6th term of a sequence. Remember that a sequence is a function which is defined in the set of natural numbers, and 2.6 is not a natural number. Clearly, expressions like a_0, a_{-1} are also meaningless. We say that such terms are **undefined**.

Note
In a sequence, n should always be a natural number, but the value of a_n may be any real number depending on the formula for the general term of the sequence.

Example 1 Write the first five terms of the sequence with general term $a_n = \frac{1}{n}$.

Solution Since we are looking for the first five terms, we just recalculate the general term for $n = 1, 2, 3, 4, 5$, which gives $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$.

Example 2 Given the sequence with general term $a_n = \frac{4n-5}{2n}$, find a_5, a_{-2}, a_{100}.

Solution We just have to recalculate the formula for a_n choosing instead of n the numbers 5, –2, and 100. So $a_5 = \frac{3}{2}$, and $a_{100} = \frac{395}{200} = \frac{79}{40}$. Clearly, a_{-2} is undefined, since –2 is not a natural number.

Example 3 Find a suitable general term b_n for the sequence whose first four terms are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$.

Solution We need to find a pattern. Notice that the numerator of each fraction is equal to the term position and the denominator is one more than the term position, so we can write $b_n = \frac{n}{n+1}$.

Check Yourself 1
1. Write the first five terms of the sequence whose general term is $c_n = (-1)^n$.
2. Find a suitable general term a_n for the sequence whose first four terms are 2, 4, 6, 8.
3. Given the sequence with general term $b_n = 2n + 3$, find b_5, b_0, and b_{43}.

Answers

1. –1, 1, –1, 1, –1 2. $2n$ 3. 13, undefined, 89

2. Criteria for the Existence of a Sequence

If there is at least one natural number which makes the general term of a sequence undefined, then there is no such sequence.

Example 4 Is $a_n = \dfrac{2n+1}{n-2}$ a general term of a sequence? Why?

Solution No, because we cannot find a proper value for $n = 2$.

Example 5 Is $a_n = \sqrt{\dfrac{4-n}{2n+1}}$ a general term of a sequence? Why?

Solution Note that the expression $\sqrt{x}$ is only meaningful when $x \geq 0$. So we need $\dfrac{4-n}{2n+1} \geq 0$ to be true for any natural number n. If we solve this equation for n, the solution set is $(-\frac{1}{2}, 4]$, i.e. n is between $-\frac{1}{2}$ and 4, inclusive. When we take the natural numbers in this solution set, we get $\{1, 2, 3, 4\}$, which means that only a_1, a_2, a_3, a_4 are defined. So a_n is not the general term of a sequence.

Example 6 Is $a_n = \dfrac{n+1}{2n-1}$ a general term of a sequence? If yes, find $a_1 + a_2 + a_3$.

Solution $\dfrac{n+1}{2n-1}$ is not meaningful only when $n = \frac{1}{2} \notin \mathbb{N}$. Since a_n is defined for any natural number, it is the general term of a sequence. Choosing $n = 1, 2, 3$ we get $a_1 = 2$, $a_2 = 1$, $a_3 = 0.8$. So $a_1 + a_2 + a_3 = 3.8$.

Example 7 Given $b_n = 2n + 5$, find the term of the sequence (b_n) which is equal to

a. 25 b. 17 c. 96

Solution

a.
$$b_n = 25$$
$$2n + 5 = 25$$
$$n = 10$$
10th term

b.
$$b_n = 17$$
$$2n + 5 = 17$$
$$n = 6$$
6th term

c.
$$b_n = 96$$
$$2n + 5 = 96$$
$$n = 45.5 \notin \mathbb{N}$$
not a term

Check Yourself 2

1. Is $a_n = \dfrac{3n+1}{n+2}$ a general term of a sequence? Why?

2. For which values of a is $b_n = \sqrt{n^2 + a}$ general term of a sequence?

3. Which term of the sequence with general term $a_n = \dfrac{3n-1}{5n+7}$ is $\dfrac{7}{12}$?

Answers

1. yes, because a_n is defined for all $n \in N$ 2. $a \in [-1, \infty)$ 3. 61^{st}

Example 8 How many terms of the sequence with general term $a_n = \dfrac{n^2-6n-7}{3n-2}$ are negative?

Solution We are looking for the number of values of n for which $a_n < 0$. In other words we should find the solution set for $\dfrac{n^2-6n-7}{3n-2} < 0$ in the set of natural numbers. Solving the inequality, we get $(-\infty, -1) \cup (\dfrac{2}{3}, 7)$. The natural numbers in this solution set are 1, 2, 3, 4, 5, and 6. Therefore, six terms of this sequence are negative.

B. TYPES OF SEQUENCE

1. Finite and Infinite Sequences

A sequence may contain a finite or infinite number of terms.

For example the sequence $(a_n) = (1, 4, 9, \ldots, n^2)$ contains n terms, which is a finite number of terms. The sequence $(b_n) = (1, 4, 9, \ldots, n^2, \ldots)$ contains infinitely many terms.

If a sequence contains a countable number of terms, then we say it is a **finite sequence**.

If a sequence contains infinitely many terms, then we say it is an **infinite sequence**.

Example 9 State whether the following sequences are finite or infinite.

a. The sequence of all odd numbers.

b. $(a_n) = (-10, -5, 0, 5, 10, 15, \ldots, 150)$

c. 1, 1, 2, 3, 5, 8, ...

Solution a. The sequence of all odd numbers is 1, 3, 5, 7, ...

Since there are infinitely many numbers here, the sequence is infinite.

b. This sequence has a finite number of terms since the last term (150) is given.

c. The sequence is infinite, as the '...' notation shows that there are infinitely many numbers.

Note

In this book, if we do not say a sequence is finite, then it is an infinite sequence.

2. Monotone Sequences

If each term of a sequence is greater than the previous term, then the sequence is called an **increasing sequence**.

Symbolically, (a_n) is an increasing sequence if $a_{n+1} > a_n$.

If $a_{n+1} \geq a_n$, then (a_n) is a **nondecreasing sequence**.

If each term of a sequence is less than the previous term, then that sequence is called a **decreasing sequence**.

Symbolically (a_n) is a decreasing sequence if $a_{n+1} < a_n$.

If $a_{n+1} \leq a_n$, then (a_n) is a **nonincreasing sequence**.

In general any increasing, nondecreasing, decreasing, or nonincreasing sequence is called a **monotone sequence**.

For example, the sequence 10, 8, 6, 4, ... is a decreasing sequence since each consecutive term is less than the previous one. Therefore, it is a monotone sequence.

The sequence 1, 1, 2, 3, 5, ... is a nondecreasing sequence, because the first two terms are equal. It is also a monotone sequence.

Consider the sequence 4, 1, 0, 1, 4, ... Obviously we cannot put this sequence into any of the categories of sequence defined above. Therefore, it is not monotone.

Note

We can rewrite the above criteria for increasing and decreasing sequences in a different way:
If $a_{n+1} - a_n > 0$, then we have an increasing sequence.
If $a_{n+1} - a_n < 0$, then we have a decreasing sequence.

Example 10 Prove that the sequence (a_n) with general term $a_n = 2n$ is an increasing sequence.

Solution If $a_n = 2n$, then $a_{n+1} = 2(n+1) = 2n+2$, and so $a_{n+1} - a_n = 2n + 2 - 2n = 2$. Since $2 > 0$, (a_n) is an increasing sequence.

Example 11 Prove that the sequence with general term $b_n = \dfrac{1}{n+1}$ is a decreasing sequence.

Solution If $b_n = \dfrac{1}{n+1}$, then $b_{n+1} = \dfrac{1}{n+2}$.

$$b_{n+1} - b_n = \frac{1}{n+2} - \frac{1}{n+1} = \frac{-1}{(n+1)(n+2)}$$

Since n is a natural number, $n+1 > 0$ and $n+2 > 0$. That means $b_n = \dfrac{-1}{(n+1)(n+2)} < 0$. Therefore, (b_n) is a decreasing sequence.

Example 12 Given the sequence with general term $a_n = -n^2 + 8n - 3$,

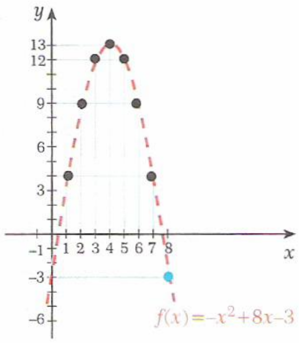
a. find the biggest term.

b. state whether the sequence is monotone or not.

Solution

The peak point of a parabola given by $f(x) = ax^2 + bx + c$ is $\left(-\dfrac{b}{2a}, f(-\dfrac{b}{2a})\right)$.

a. If we think about the general term in functional notation, we have $f(x) = -x^2 + 8x - 3$, whose graph is the parabola shown opposite.



Here, note that we cannot talk about a minimum value. Clearly, the parabola takes its maximum value at its peak point and so does the sequence, provided that the x-coordinate at that peak point is a natural number. The peak point of the parabola lies at $x = \dfrac{-8}{-2} = 4$. Since $4 \in \mathbb{N}$, the biggest term of the sequence is $f(4) = a_4 = 13$. (What would you do if the x-coordinate at the peak point was not a natural number?)

b. If we look at the above parabola's values for natural values of x (the black dots), we can see that the sequence is increasing before $x = 4$ and then decreasing. Therefore, the sequence cannot be defined as increasing or decreasing, which means that it is not monotone.

Check Yourself 3

1. State if the following sequences are finite or infinite.

   a. The sequence with general term $c_n = \dfrac{1}{n+1}$.

   b. 3, 6, 9, ..., 54

   c. 3, 6, 9, ...

2. Prove that $(a_n) = (2 - 5n)$ is a decreasing sequence.

3. Classify the following sequences as increasing or decreasing.

   a. $(a_n) = (2n + 1)$ b. $(b_n) = (\dfrac{4}{n})$ c. $(c_n) = (\dfrac{n+1}{2n-8})$ d. $(d_n) = (n^2 - 4n)$

4. For which term(s) does the sequence $(c_n) = (n^2 - 5n + 7)$ take its minimum value?

   Hint: Consider the nearest natural x-coordinates to the minimum of the graph of $f(x) = (x^2 - 5x + 7)$.

Answers

1. a. infinite b. finite c. infinite 3. a. increasing b. decreasing c. not a sequence d. neither

4. $n = 2$ and $n = 3$, i.e. the second and third terms

3. Piecewise Sequences

If the general term of a sequence is defined by more than one formula, then it is called a **piecewise sequence**.

For example, the sequence with general term

$$a_n = \begin{cases} \dfrac{1}{n} & , n \text{ is even} \\ \dfrac{2}{n+1}, & n \text{ is odd} \end{cases}$$

is a piecewise sequence.

Example 13 Write the first four terms of the piecewise sequence with general term $a_n = \begin{cases} \dfrac{1}{n} & , n \text{ is even} \\ \dfrac{2}{n+1}, & n \text{ is odd} \end{cases}$.

Solution To find a_1 and a_3 we use $\dfrac{1}{n}$ since n is odd, and to find a_2 and a_4 we use $\dfrac{2}{n+1}$ since n is even.

So $a_1 = 1$, $a_2 = \dfrac{2}{3}$, $a_3 = \dfrac{1}{3}$, and $a_4 = \dfrac{2}{5}$.

Example 14 Given the piecewise sequence with general term $a_n = \begin{cases} n^2 - 5n & , n < 10 \\ n - 8 & , n \geq 10 \end{cases}$,

a. find a_{20}.

b. find a_1.

c. find the term which is equal to 0.

Solution a. When $n = 20$, $a_n = n - 8$. So $a_{20} = 20 - 8 = 12$.

b. When $n = 1$, $a_n = n^2 - 5n$. So $a_1 = 1^2 - 5 \cdot 1 = -4$.

c. If a term is equal to 0, then $a_n = 0$. This means

$$n^2 - 5n = 0 \quad (\text{for } n < 10) \qquad \text{or} \qquad n - 8 = 0 \quad (\text{for } n \geq 10)$$

$$n(n-5) = 0 \qquad\qquad n = 8 \ngeq 10$$

$$n = 0 \notin \mathbb{N} \quad \text{or} \quad n = 5$$

So $a_5 = 0$.

4. Recursively Defined Sequences

Sometimes the terms in a sequence may depend on the other terms. Such a sequence is called a **recursively defined sequence**.

For example, the sequence given with general term $a_{n+1} = a_n + 3$ and first term $a_1 = 4$ is a recursively defined sequence.

Example 15 Given $a_1 = 4$ and $a_{n+1} = a_n + 3$,

a. find a_2.

b. find the general term of the sequence.

Solution a. Note that choosing $n = 2$ will not help us to find a_2 since we will get an equation like $a_3 = a_2 + 3$, which needs a_3 to get a_2.

But if we choose $n = 1$, we will get $a_2 = a_1 + 3$. Using $a_1 = 4$, we find $a_2 = 4 + 3 = 7$.

b. $a_2 = a_1 + 3$

$a_3 = a_2 + 3 = a_1 + 3 + 3$

$a_4 = a_3 + 3 = a_1 + 3 + 3 + 3$

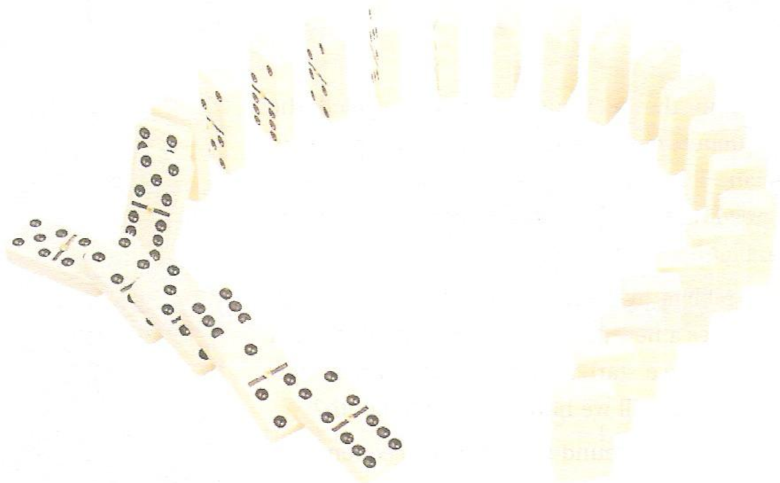
$a_5 = a_4 + 3 = a_1 + 3 + 3 + 3 + 3$

$\vdots$

$a_n = a_1 + (n-1) \cdot 3$

$a_n = 4 + (n-1) \cdot 3$

So the general term is $a_n = 3n + 1$.



Recursively defined sequences have terms which depend on previous ones like the falling dominoes above.

Example 16 Given $f_1 = 1, f_2 = 1, f_n = f_{n-2} + f_{n-1}$ (for $n \geq 3$), find the first six terms of the sequence.

Solution When we consider the general term, we notice that it is not possible to calculate a term's value unless we know the two previous terms. Since we are given the first and second terms, with the help of the general term we can find the third term.

Choosing $n = 3$, the formula for general term becomes $f_3 = f_1 + f_2 = 1 + 1 = 2$. Now it is possible to find a_4, and then by the same procedure a_5 and a_6.

$f_4 = f_2 + f_3 = 1 + 2 = 3$

$f_5 = f_3 + f_4 = 2 + 3 = 5$

$f_6 = f_4 + f_5 = 3 + 5 = 8$

The first six terms are 1, 1, 2, 3, 5, 8.

Since recursively defined sequences have terms which depend on previous ones like a chain, we calculate the terms one by one to find the desired term. In the above example, unless we find a direct formula for the general term (is it possible?), it will take too much time and effort to find f_{1000}.

THE FIBONACCI SEQUENCE AND THE GOLDEN RATIO

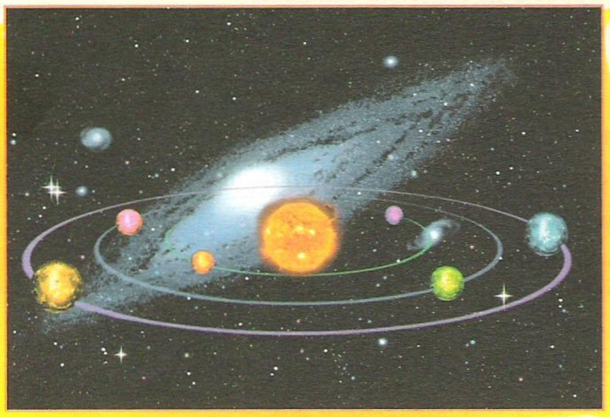
The sequence in the previous example is called the **Fibonacci sequence**, named after the 13th century Italian mathematician Fibonacci, who used it to solve a problem about the breeding of rabbits. Fibonacci considered the following problem:

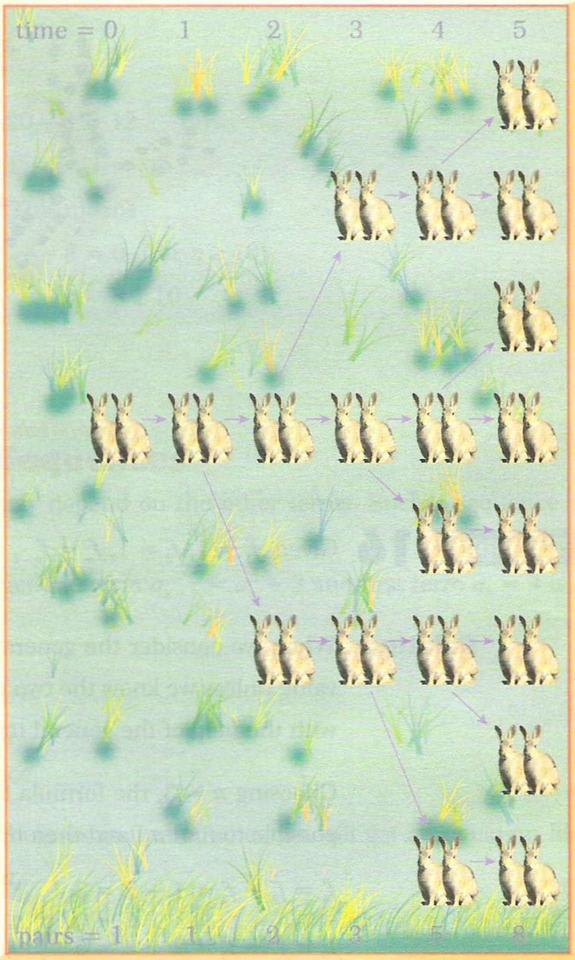
Suppose that rabbits live forever and that every month each pair produces a new pair that becomes productive at age two months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month?

As a solution, Fibonacci found the following sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

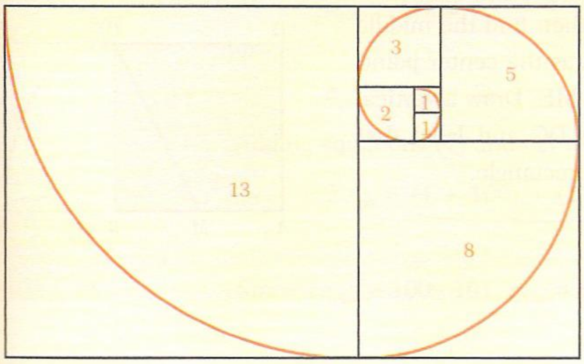
This sequence also occurs in numerous other aspects of the natural world.





The planets in our solar system are spaced in a Fibonacci sequence.

We can make a picture showing the Fibonacci numbers if we start with two small squares whose sides are each one unit long next to each other. Then we draw a square with side length two units (1 + 1 units) next to both of these. We can now draw a new square which touches the square with side one unit and the square with side two units, and therefore has side three units. Then we draw another square touching the two previous squares (side five units), and so on. We can continue adding squares around the picture, each new square having a side which is as long as the sum of the sides of the two previous squares. Now we can draw a spiral by connecting the quarter circles in each square, as shown on the next page. This is a spiral (the **Fibonacci Spiral**). A similar curve to this occurs in nature as the shape of a nautilus.



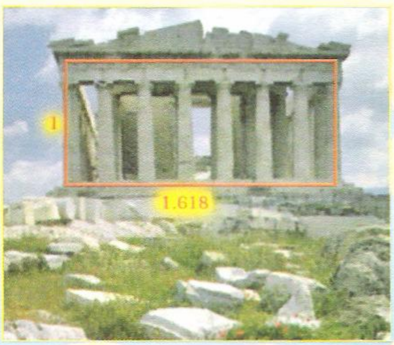
A nautilus has the same shape as the Fibonacci spiral.

The ratio of two successive Fibonacci numbers $\frac{f_{n+1}}{f_n}$ gets closer to the number $\frac{1+\sqrt{5}}{2} \approx 1.618$ as the value of n gets bigger. This number is a special number in mathematics and is known as the **golden ratio**.

The ancient Greeks also considered a line segment divided into two parts such that the ratio of the shorter part of length one unit to the longer part is the same as the ratio of the longer part to the whole segment.

This leads to the equation $\frac{1}{x} = \frac{x}{1+x}$ whose positive solution is $x = \frac{1+\sqrt{5}}{2}$. Thus, the segment shown is divided into the golden ratio!

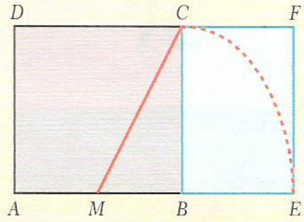
A rectangle in which the ratio of one side to the other gives the golden ratio is called a **golden rectangle**. The Golden Rectangle is a unique and a very important shape in mathematics. It appears in nature and music, and is also often used in art and architecture. The Golden Rectangle is believed to be one of the most pleasing and beautiful shapes for the human eye.



The golden ratio is frequently used in architecture.

The ratio of the length of your arm to the length from the elbow down to the end of your hand is approximately equal to the golden ratio.

To construct a golden rectangle, draw a square ABCD and then find the middle point M of the base AB. Draw a line from M to C. Using M as the center point, rotate the line MC until it overlaps AB. Name this new line ME. Draw a vertical up from point E until it intersects the extension of line DC and label that intersection as point F. The new rectangle AEFD is a golden rectangle.



Example 17 Given $a_5 = 6$ and $(n + 2) \cdot a_{n-1} = 3a_n$ (for $n \geq 2$), find a_3.

Solution This time we are given the fifth term and the third term is required. This means we should think backwards. That is, first we should find a_4 and then a_3.

Choosing $n = 5$, the formula for general term becomes $7a_4 = 3a_5$, i.e. $a_4 = \frac{18}{7}$. Now it is possible to find a_3 by choosing $n = 4$: $6a_3 = 3a_4$, so $a_3 = \frac{9}{7}$.

Example 18 Given $a_1 = 1$ and $a_n = a_{n-1} + n$ (for $n \geq 2$), find a_{100}.

Solution Since we are given a recursively defined sequence, it will take too much effort to find the hundredth term unless we find a more practical way. Let us write a few terms:

Clearly,
$$\begin{aligned} a_1 &= 1 \\ a_2 &= a_1 + 2 \\ a_3 &= a_2 + 3 \\ a_4 &= a_3 + 4 \\ &\vdots \\ a_{99} &= a_{98} + 99 \\ a_{100} &= a_{99} + 100 \end{aligned}$$

If we add each side of the equations, we get

$$a_1+a_2+a_3+a_4 + \ldots +a_{99}+a_{100} = a_1+a_2+a_3+ \ldots +a_{98}+a_{99}+1+2+3+4+ \ldots +99+100,$$

which we can simplify as

$$a_{100} = 1 + 2 + 3 + 4 + \ldots + 99 + 100 \quad (1)$$

or

$$a_{100} = 100 + 99 + \ldots + 4 + 3 + 2 + 1. \quad (2)$$

Adding equations (1) and (2) we get

$$2a_{100} = \underbrace{(1 + 100) + (2 + 99) + \ldots + (99 + 2) + (100 + 1)}_{100 \text{ terms}}$$

Since $2a_{100} = 100 \cdot 101$, $a_{100} = 5050$.

Recursively defined sequences are frequently used in computer programming.

Their disadvantage is that we cannot find any term directly, but their advantage is that we can successfully model more complicated systems as we saw for Fibonacci's problem.

Check Yourself 4

1. Given $a_n = \begin{cases} 2n + 1 & , \quad n < 6 \\ n^2 - 1 & , \quad 6 \le n \le 13, \\ 4 & , \quad n > 13 \end{cases}$

   find the biggest and smallest terms of the sequence.

2. Given $a_1 = 1$, $a_2 = \dfrac{1}{2}$ and $a_n = a_{n+1} - a_{n-1}$ (for $n \ge 2$), find a_5.

3. Given $a_1 = 1$ and $a_n = 2a_{n-1} + 1$ (for $n \ge 2$), which term of the sequence is equal to 63?

Answers

1. a_{13} biggest, a_1 smallest 2. 3.5 3. 6th

EXERCISES 1

A. Sequences

1. State whether each term is a general term of a sequence or not.

a. $3n - 76$ b. $\dfrac{n}{n+2}$ c. $\dfrac{2n+1}{2n-1}$

d. $\dfrac{4}{n^2-4}$ e. $\dfrac{13}{4}$ f. $(-1)^n \dfrac{1}{n^3}$

g. $\sqrt{n-5}$ h. $\sqrt{n^2+2n}$ i. $\sqrt{\dfrac{n^2-n-2}{n-2}}$

2. Find a suitable formula for the general terms of the sequences whose first few terms are given.

a. 1, 3, 5 b. –1, 3, –5

c. 0, 3, 8, 15 d. $-\dfrac{1}{5}, -\dfrac{8}{7}, -\dfrac{27}{9}$

e. 2, 6, 12, 20, 30

3. Find the stated terms for the sequence with the given general term.

a. $a_n = 2n + 3$, find the first three terms and a_{37}

b. $a_n = \dfrac{3n+1}{n+7}$, find the first three terms and a_{33}

c. $a_n = \sqrt{n^2+6n}$, find the first three terms and a_6

4. How many terms of the sequence with general term $a_n = n^2 - 6n - 16$ are negative?

5. How many terms of the sequence with general term $a_n = \dfrac{3n-7}{3n+5}$ are less than $\dfrac{1}{5}$?

6. For the sequence with general term

$a_n = \dfrac{n^2-2n}{1-k+n}$ and $a_5 = 5$, find k.

7. Find a suitable general term (not piecewise) for the sequence whose first five terms are 2, 4, 6, 8, 34. What is the sixth term?

B. Types of Sequence

8. For the sequence with general term

$$a_n = \begin{cases} 2n + 1 & , \quad n \text{ even} \\ n^2 & , \quad n \text{ odd} \end{cases}$$

find $a_4 + a_7$.

9. Find a suitable general term for the sequence whose first six terms are 2, 1, 4, 3, 6, 5.

10. Prove that the sequence with general term

a. $a_n = 4n - 17$ is increasing.

b. $b_n = 25 \cdot (\dfrac{1}{5})^n$ is decreasing.

11. State whether the sequence $b_n = \dfrac{3n-7}{n+2}$ is monotone or not.

12. Find the biggest and smallest terms (if they exist) of the sequences with the following general terms.

a. $a_n = |3n - 5|$ b. $b_n = -n^2 + 4n + 7$

c. $c_n = \dfrac{3n-5}{2n+1}$

13. Find the first four terms and, if possible, the general term of the recursively defined sequences.

a. $a_1 = 1,\ a_{n+1} = 2a_n$

b. $b_1 = -3,\ b_{n+1} = 5 + b_n$

c. $a_1 = 3,\ a_n = (2n + 1)a_{n-1}$

14. Write the following sequences recursively.

a. $a_n = 3n$ b. $b_n = 2^n$

c. $c_n = 8 \cdot (-\frac{1}{2})^n$

15. Given a sequence with $a_{n+1} = \dfrac{2a_n + 3}{2}$ and $a_1 = 3$, find a_{29}.

16. Consider a sequence with $a_{n+1} = \dfrac{n+2}{n} \cdot a_n$ and $a_1 = 2$. Is 1980 a term of this sequence?

Mixed Problems

17. Given the sequences with general terms $a_n = (-2)^n + 2$, $b_n = 4 + 4^n$, $c_n = 2 - (-2)^n$, find d_{2003} where $d_n = a_n \cdot b_n \cdot c_n + (-4)^{2n}$.

18. Given the sequence with general term $a_n = 5^n \cdot n!$, find $\dfrac{a_n}{a_{n-1}}$.

$\left(n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \text{ where } n \in \mathbb{N} \right)$

19. Given the sequence with general term $a_n = \dfrac{(n+1)!}{3^n}$ find $\dfrac{a_{n+1}}{a_n}$.

20. How many terms of the sequence with general term $a_n = \dfrac{3n - 72}{n}$ are integers?

21. How many terms of the sequence with general term $a_n = \dfrac{n^3 + 4n^2 + 3n + 1}{n+2}$ are integers?

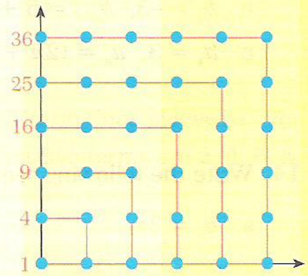
22. Find the greatest integer b for which the sequence with general term $a_n = \dfrac{bn - 3}{-3n - 2}$ is increasing.

23. Find all values of p for which the sequence with general term $c_n = \dfrac{2003n + p}{2004}$ is increasing.

24. The sequence (f_n) where $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$ is known as the **Fibonacci sequence.** Prove that $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = f_n \cdot f_{n+1}$.

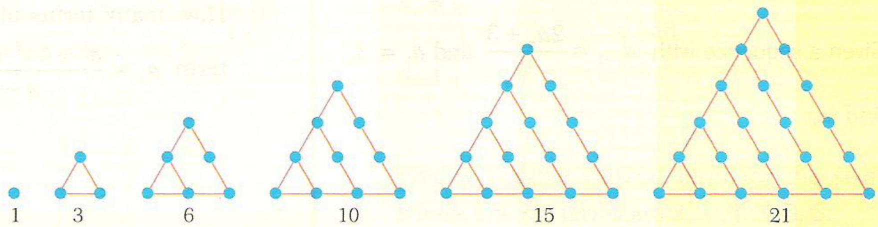
POLYGONAL NUMBERS

At the beginning of this book we looked at the sequence 1, 4, 9, 16, 25, 36, ... We call the numbers in this sequence **square numbers**. We can generate the square numbers by creating a sequence of nested squares like the one on the right. Starting from a common vertex, each square has sides one unit longer than the previous square. When we count the number of points in each successive square, we get the sequence of square numbers
(first square = 1 point, second square = 4 points, third square = 9 points, etc.).

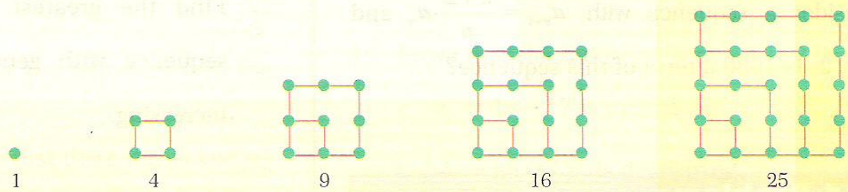


Polygonal numbers are numbers which form sequences like the one above for different polygons. The Pythagoreans named these numbers after the polygons that defined them.

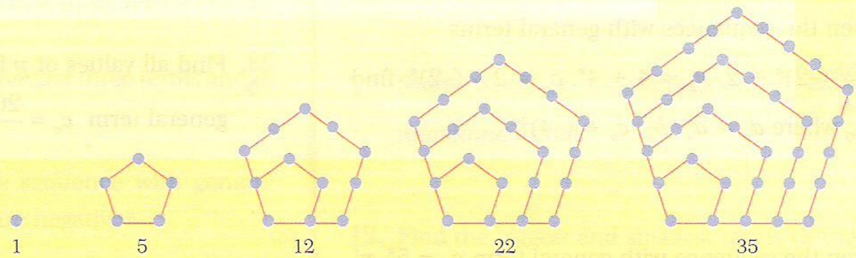
Triangular numbers



Square numbers



Pentagonal numbers



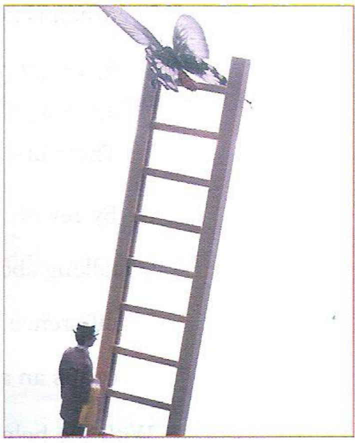
Polygonal numbers have many interesting relationships between them. For example, the sum of any two consecutive triangular numbers is a square number, and eight times any triangular number plus one is always a square number.

Can you find any more patterns? Can you find the general term for each set of polygonal numbers?

2 ARITHMETIC SEQUENCES

A. ARITHMETIC SEQUENCES

1. Definition



Let's look at the sequence 6, 10, 14, 18, …

Obviously the difference between each term is equal to 4 and the sequence can be written as $a_{n+1} = a_n + 4$ where $a_1 = 6$.

For the sequence 23, 21, 19, … the formula will be $a_{n+1} = a_n - 2$ where $a_1 = 23$.

In these examples, the difference between consecutive terms in each sequence is the same. We call sequences with this special property **arithmetic sequences**.

Definition — arithmetic sequence

If a sequence (a_n) has the same difference d between its consecutive terms, then it is called an **arithmetic sequence**.

In other words, (a_n) is arithmetic if $a_{n+1} = a_n + d$ such than $n \in \mathbb{N}$, $d \in \mathbb{R}$. We call d the **common difference** of the arithmetic sequence. In this book, from now on we will use a_n to denote general term of an arithmetic sequence and d (the first letter of the Latin word *differentia*, meaning difference) for the common difference.

If d is positive, we say the arithmetic sequence is **increasing**. If d is negative, we say the arithmetic sequence is **decreasing**. What can you say when d is zero?

EXAMPLE 19 State whether the following sequences are arithmetic or not. If a sequence is arithmetic, find the common difference.

a. 7, 10, 13, 16, … b. 3, –2, –7, –12, … c. 1, 4, 9, 16, … d. 6, 6, 6, 6, …

Solution a. arithmetic, $d = 3$ b. arithmetic, $d = -5$ c. not arithmetic d. arithmetic, $d = 0$

EXAMPLE 20 State whether the sequences with the following general terms are arithmetic or not. If a sequence is arithmetic, find the common difference.

a. $a_n = 4n - 3$ b. $a_n = 2^n$ c. $a_n = n^2 - n$ d. $a_n = \dfrac{n^2+5n+4}{n+4}$

Solution a. $a_{n+1} = 4(n + 1) - 3 = 4n + 1$, so the difference between each consecutive term is $a_{n+1} - a_n = (4n + 1) - (4n - 3) = 4$, which is constant. Therefore, (a_n) is an arithmetic sequence and $d = 4$.

b. $a_{n+1} = 2^{n+1}$, so the difference between each consecutive term is $a_{n+1} - a_n = 2^{n+1} - 2^n = 2^n$, which is not constant. Therefore, (a_n) is not an arithmetic sequence.

c. $a_{n+1} = (n + 1)^2 - (n + 1)$, so the difference between two consecutive terms is $a_{n+1} - a_n = [(n + 1)^2 - (n + 1)] - (n^2 - n) = 2n$, which is not constant. Therefore, (a_n) is not an arithmetic sequence.

d. By rewriting the general term we have $a_n = \dfrac{(n+4)(n+1)}{n+4}$. Since $n \neq -4$ (since we are talking about a sequence), we have $a_n = n + 1$. Therefore, $a_{n+1} = (n + 1) + 1$, and the difference between the consecutive terms is $a_{n+1} - a_n = 1$, which is constant. Therefore, (a_n) is an arithmetic sequence and $d = 1$.

With the help of the above example we can notice that if the formula for general term of a sequence gives us a linear function, then it is arithmetic.

Note
The general term of an arithmetic sequence is linear.


Arithmetic growth is linear.

2. General Term

Since arithmetic sequences have many applications, it is much better to express the general term directly, instead of recursively. The formula is derived as follows:

If (a_n) is arithmetic, then we only know that $a_{n+1} = a_n + d$. Let us write a few terms.

$a_1 = a_1$

$a_2 = a_1 + d$

$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$

$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$

$a_5 = a_1 + 4d$

$\vdots$

$a_n = a_1 + (n - 1)d$

This is the general term of an arithmetic sequence.

GENERAL TERM FORMULA

The general term of an arithmetic sequence (a_n) with common difference d is

$$a_n = a_1 + (n - 1)d.$$

EXAMPLE 21 $-3, 2, 7$ are the first three terms of an arithmetic sequence (a_n). Find the twentieth term.

Solution We know that $a_1 = -3$ and $d = a_3 - a_2 = a_2 - a_1 = 5$. Using the general term formula,

$$a_n = a_1 + (n - 1)d$$

$$a_{20} = -3 + (20 - 1) \cdot 5 = 92.$$

EXAMPLE 22 (a_n) is an arithmetic sequence with $a_1 = 4$, $a_8 = 25$. Find the common difference and a_{101}.

Solution Using the general term formula,

$$a_n = a_1 + (n - 1)d$$

$$a_8 = a_1 + 7d$$

$25 = 4 + 7d$. So we have $d = 3$.

$$a_{101} = a_1 + (100 - 1)d = 4 + 100 \cdot 3 = 304$$

EXAMPLE 23 (a_n) is an arithmetic sequence with $a_1 = 3$ and common difference 4. Is 59 a term of this sequence?

Solution For 59 to be a term of the arithmetic sequence, it must satisfy the general term formula such that n is a natural number.

$$a_n = a_1 + (n - 1)d$$

$$59 = 3 + (n - 1) \cdot 4$$

$$59 = 4n - 1$$

$$n = 15$$

Since 15 is a natural number, 59 is the 15th term of this sequence.

EXAMPLE 24 Find the number of terms in the arithmetic sequence 1, 4, 7, ..., 91.

Solution Here we have a finite sequence. Using the general term formula,

$$a_n = a_1 + (n - 1)d$$

$$91 = 1 + (n - 1) \cdot 3$$

$$n = 31$$

Therefore, this sequence has 31 terms.

Note that if we rewrite the general term formula in terms of n, we get $n=\frac{a_n-a_1}{d}+1$, which is the number of terms in a finite arithmetic sequence.

NUMBER OF TERMS OF A FINITE ARITHMETIC SEQUENCE

The number of terms in a finite arithmetic sequence is $n=\frac{a_n-a_1}{d}+1$, where a_1 is the first term, a_n is the last term, and d is the common difference.

EXAMPLE 25 How many two-digit numbers are divisible by 5?

Solution These numbers form a finite arithmetic sequence since the number of two-digit numbers is finite, and the difference between consecutive numbers in this sequence is constant, that is 5. We have $a_1 = 10$ (the smallest two-digit number divisible by 5), and $a_n = 95$ (the greatest two-digit number divisible by 5).

Therefore, $n=\frac{a_n-a_1}{d}+1=\frac{95-10}{5}+1=18$.

Therefore, 18 two-digit numbers are divisible by 5.

Check Yourself 5

1. Is the sequence with general term $a_n = 5n + 9$ an arithmetic sequence? Why?
2. 6, 2, –2 are the first three terms of an arithmetic sequence (a_n). Find the 30th term.
3. (a_n) is an arithmetic sequence with $a_1 = 7$, $a_{10} = 70$. Find the common difference and a_{101}.
4. (a_n) is an arithmetic sequence with $a_1 = -1$ and common difference 9. Which term of this sequence is 89?
5. How many three-digit numbers are divisible by 30?

Answers

1. yes; linear formula 2. –110 3. 7; 707 4. 11th 5. 30

3. Advanced General Term Formula

EXAMPLE 26 (a_n) is an arithmetic sequence with $a_{11} = 34$ and common difference 3. Find a_3.

Solution Using the general term formula,

$a_n = a_1 + (n-1)d$
$a_{11} = a_1 + (11-1)\cdot 3$
$34 = a_1 + 30$
$a_1 = 4$
$a_3 = a_1 + 2d = 4 + 6$, so $a_3 = 10$.

In this example, we calculated the first term of the sequence (a_1) from a_{11}, then used this value to find a_3. However, there is a quicker way to solve this problem: in general, if we know the common difference and any term of an arithmetic sequence, we can find the required term without finding the first term. Look at the calculation:

If we know a_p and d, to find a_n we can write:

$$a_n = a_1 + (n - 1)d \qquad (1)$$

$$a_p = a_1 + (p - 1)d \qquad (2)$$

Subtracting (2) from (1), we get $a_n - a_p = (n - p)d$.

So $a_n = a_p + (n - p)d$.

ADVANCED GENERAL TERM FORMULA

The general term of an arithmetic sequence (a_n) with common difference d is $a_n = a_p + (n - p)d$, where a_p is any term of that sequence.

So using the advanced general term formula, we can solve the previous example as follows:

$$a_n = a_p + (n - p)d$$

$$a_{11} = a_3 + (11 - 3) \cdot 3$$

$$34 = a_3 + 24$$

$$a_3 = 10.$$

Here it is not important which term you write in the place of a_n and a_p.

Note that when $p = 1$, the advanced general term formula becomes the general term formula we studied previously.

EXAMPLE 27 (a_n) is an arithmetic sequence with $a_5 = 14$ and $a_{10} = 34$. Find the common difference.

Solution Using the advanced general term formula,

$$a_n = a_p + (n - p)d$$

$$a_{10} = a_5 + (10 - 5)d$$

$$34 = 14 + 5d$$

$$d = 4.$$

EXAMPLE 28 (a_n) is an arithmetic sequence with $a_9 - a_2 = 42$. Find $a_{10} - a_7$.

Solution Using the advanced general term formula,

$$a_9 = a_2 + 7d$$

$$a_9 - a_2 = 7d$$

$$42 = 7d$$

$$d = 6.$$

Therefore, $a_{10} = a_7 + 3d$

$$a_{10} - a_7 = 3 \cdot 6 = 18.$$

EXAMPLE 29 $4, x, y, z,$ and 24 are five consecutive terms of an arithmetic sequence. Find x, y, and z.

Solution Let $a_p = 4$, then $a_{p+4} = 24$. Using the advanced general term formula,

$$a_{p+4} = a_p + (p + 4 - p)d$$
$$24 = 4 + 4d$$
$$d = 5.$$

Since the difference between consecutive terms is 5,

$$x = 4 + 5 = 9,$$
$$y = 9 + 5 = 14,$$
$$z = 14 + 5 = 19.$$

EXAMPLE 30 We insert five numbers in increasing order between 12 and 42 such that all the numbers form an arithmetic sequence. Find the third number of this sequence.

Solution If we begin with two numbers and insert five numbers, the sequence has seven numbers in total. Let us call the first number a_1, the second a_2, and so on. We can now write the problem differently: given an arithmetic sequence (a_n) with $a_1 = 12$, $a_7 = 42$, find a_3.

> The common difference of an arithmetic sequence formed by inserting k terms between two real numbers b and c is
>
> $d = \dfrac{c-b}{k+1}.$

Using the general term formula,

$$a_7 = a_1 + 6d$$
$$42 = 12 + 6d$$
$$d = 5$$
$$a_3 = a_1 + 2d$$
$$a_3 = 12 + 10$$
$$a_3 = 22.$$

EXAMPLE 31 Given an arithmetic sequence (a_n) with $a_8 = 10$, find $a_2 + a_{14}$.

Solution This time we have just $a_8 = 10$ as data. Until now we have learned just one fundamental formula $a_n = a_1 + (n - 1)d$, and the advanced general term formula we derived from it. We cannot find a_2 or a_{14} with the help of the general term formula since we need two values as data. However, remember that we are not asked to find a_2 or a_{14}, but to find $a_2 + a_{14}$. Let's apply the advanced general term formula, keeping in mind that we just know a_8:

$$a_2 = a_8 + (2 - 8)d \qquad (1)$$
$$a_{14} = a_8 + (14 - 8)d. \qquad (2)$$

Adding equations (1) and (2) we get

$a_2 + a_{14} = a_8 - 6d + a_8 + 6d = 2a_8 = 20.$

4. Middle Term Formula (Arithmetic Mean)

The solution to the previous example shows us a practical formula.

Let a_p and a_k be terms of an arithmetic sequence such that $k < p$. Then

$$a_{p-k} = a_p - kd \qquad (1)$$

$$a_{p+k} = a_p + kd. \qquad (2)$$

Adding equations (1) and (2) we get

$a_{p-k} + a_{p+k} = 2a_p$, or $a_p = \dfrac{a_{p-k} + a_{p+k}}{2}$, which means that any term x in an arithmetic sequence is half the sum of any two terms which are at equal distance from x in the sequence.

Note that in the previous example, a_8 was at equal distance from a_2 and a_{14}. (Could we solve the problem if we were given not a_8 but a_{10}?)

MIDDLE TERM FORMULA (Arithmetic Mean)

In an arithmetic sequence, $a_p = \dfrac{a_{p-k} + a_{p+k}}{2}$ where $k < p$.

The arithmetic mean (or average) of two numbers x and y is m:

$$m = \frac{x+y}{2}$$

Note that m is the same distance from x as from y so x, m, y form a finite arithmetic sequence.

For example, all the following equalities will hold in an arithmetic sequence:

$a_2 = \dfrac{a_1 + a_3}{2}$ since a_2 is in the middle of a_1 and a_3

$a_6 = \dfrac{a_5 + a_7}{2} = \dfrac{a_1 + a_{11}}{2} = \dfrac{a_4 + a_x}{2}$ (x must be 8)

$\dfrac{a_{12} + a_{20}}{2} = a_y$ (y must be 16)

EXAMPLE 32 5, x, 19 are three consecutive terms of an arithmetic sequence. Find x.

Solution If we say $a_1 = 5$, $a_2 = x$, $a_3 = 19$, then using the middle term formula,

$a_2 = \dfrac{a_1 + a_3}{2}$ and $x = \dfrac{5+19}{2} = 12$. Therefore, x is 12 if the sequence is arithmetic.

Note

Three numbers a, b, c form an arithmetic sequence if and only if $b = \dfrac{a+c}{2}$.

EXAMPLE 33 Find the general term a_n for the arithmetic sequence with $a_5 + a_{21} = 106$ and $a_9 = 37$.

Solution Using the middle term formula, $\frac{a_5 + a_{21}}{2} = a_{13} = \frac{106}{2}$. So $a_{13} = 53$.

Using the advanced general term formula,

$$a_{13} = a_9 + 4d$$
$$53 = 37 + 4d$$
$$d = 4.$$

To write the general term we can choose a_9 or a_{13}. Let us choose a_9, then using the advanced general term formula we get

$$a_n = a_9 + (n - 9)d$$
$$a_n = 37 + (n - 9) \cdot 4$$
$$a_n = 4n + 1.$$

Check Yourself 6

1. (a_n) is an arithmetic sequence with $a_{17} = 41$ and common difference -4. Find a_3.
2. (a_n) is an arithmetic sequence with $a_5 = 19$, $a_{14} = 55$. Find the common difference.
3. Fill in the blanks to form an arithmetic sequence: _, _, _, _, 2, _, _, _, 8.
4. Find x if x, 4, 19 form an arithmetic sequence.
5. Find the general term a_n for the arithmetic sequence with $a_3 + a_{19} = 98$, $d = 7$.

Answers

1. 97 2. 4 3. –2.5, –1, 0.5 and then 3.5, 5, 6.5 4. –11 5. $7n - 28$

EXAMPLE 34 Given an arithmetic sequence (a_n) with $a_1 = 100$ and a_{22} as the first negative term, how many integer values can d take?

Solution Let's convert the problem into algebraic language:

$\begin{matrix} a_1 = 100 \\ d \in \mathbb{Z} \end{matrix}$, $\begin{cases} a_{22} < 0 \\ a_{21} \geq 0 \end{cases}$ since a_{22} is the first negative term.

Since we are looking for the common difference (d), we need to express the above system of inequalities in terms of d:

$$\begin{cases} a_{22} < 0 \\ a_{21} \geq 0 \end{cases}, \text{ that is } \begin{cases} a_1 + 21d < 0 \\ a_1 + 20d \geq 0 \end{cases}, \text{ so } \begin{cases} d < -\frac{100}{21} \\ d \geq -5 \end{cases}.$$

The only integer that is in the solution set for the above inequalities is –5, so d can take only one integer value (–5).

EXAMPLE 35 Given a decreasing arithmetic sequence (a_n) with $a_2 + a_4 + a_6 = 18$ and $a_2 \cdot a_4 \cdot a_6 = -168$, find a_1 and d.

Solution We are given the system $\begin{cases} a_2 + a_4 + a_6 = 18 \\ a_2 \cdot a_4 \cdot a_6 = -168 \end{cases}$.

Since we are asked to find a_1 and d, it is more practical to express a_2, a_4, a_6 in terms of a_1 and d. This gives us:

$$\begin{cases} a_1 + d + a_1 + 3d + a_1 + 5d = 18 \\ (a_1 + d) \cdot (a_1 + 3d) \cdot (a_1 + 5d) = -168 \end{cases}, \text{ so } \begin{cases} 3a_1 + 9d = 18 & (1) \\ (a_1 + d) \cdot (a_1 + 3d) \cdot (a_1 + 5d) = -168. & (2) \end{cases}$$

From equation (1), $a_1 = 6 - 3d$.

Equation (2) becomes:

$$(6 - 2d) \cdot 6 \cdot (6 + 2d) = -168$$
$$-4d^2 + 64 = 0$$
$$d = \pm 4.$$

Since (a_n) is a decreasing arithmetic sequence, we take $d = -4$.

Finally, substituting $d = -4$ in equation (1) gives us $a_1 = 18$.

So the answer is $a_1 = 18$ and $d = -4$.

B. SUM OF THE TERMS OF AN ARITHMETIC SEQUENCE

1. Sum of the First n Terms

Let us consider an arithmetic sequence whose first few terms are 3, 7, 11, 15, 19.

The sum of the first term of this sequence is obviously 3. The sum of the first two terms is 10, the sum of the first three terms is 21, and so on. To write this in a more formal way, let us use S_n to denote **the sum of the first n terms**, i.e., $S_n = a_1 + a_2 + \ldots + a_n$. Now we can write:

$$S_1 = 3$$
$$S_2 = 3 + 7 = 10$$
$$S_3 = 3 + 7 + 11 = 21$$
$$S_4 = 3 + 7 + 11 + 15 = 36$$
$$S_5 = 3 + 7 + 11 + 15 + 19 = 55.$$

EXAMPLE 36 Given the arithmetic sequence with general term $a_n = 3n + 1$, find the sum of first three terms.

Solution $S_3 = a_1 + a_2 + a_3 = 4 + 7 + 10 = 21.$

How could we find S_{100} in the above example? Calculating terms and finding their sums takes time and effort for large sums. Since arithmetic sequences are of special interest and importance, we need a more efficient way of calculating the sums of arithmetic sequences. The following theorem meets our needs:

Theorem

The sum of the of first n terms of an arithmetic sequence (a_n) is $S_n = \dfrac{a_1 + a_n}{2} \cdot n$.

Proof $S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$ or

$S_n = a_n + a_{n-1} + \ldots + a_2 + a_1$.

Adding these equations side by side,

$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \ldots + (a_{n-1} + a_2) + (a_n + a_1)$$

$$2S_n = (a_1 + a_n) + (a_1 + d + a_n - d) + \ldots + (a_n - d + a_1 + d) + (a_n + a_1)$$

$$2S_n = \underbrace{(a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) + (a_1 + a_n)}_{\text{n times}}$$

$$2S_n = (a_1 + a_n) \cdot n$$

$$S_n = \frac{a_1 + a_n}{2} \cdot n.$$

EXAMPLE 37 Given an arithmetic sequence with $a_1 = 2$ and $a_6 = 17$, find S_6.

Solution Using the sum formula,

$$S_6 = \frac{a_1 + a_6}{2} \cdot 6 = (2 + 17) \cdot 3 = 57.$$

EXAMPLE 38 Given an arithmetic sequence with $a_1 = -14$ and $d = 5$, find S_{27}.

Solution Using the sum formula,

$S_{27} = \dfrac{a_1 + a_{27}}{2} \cdot 27$ requires $a_{27} = a_1 + 26d = -14 + 26 \cdot 5 = 116$.

Therefore, $S_{27} = \dfrac{-14 + 116}{2} \cdot 27 = 1377.$

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EXAMPLE 39 Given an arithmetic sequence with $a_1 = 56$ and $a_{11} = -14$, find S_{15}.

Solution Using the sum formula,

$S_{15} = \dfrac{a_1 + a_{15}}{2} \cdot 15$, so we need to find a_{15}. Let us calculate using a_{11}:

$a_{11} = a_1 + 10d$

$-14 = 56 + 10d$, so $d = -7$ and

$a_{15} = a_1 + 14d = 56 + 14 \cdot (-7) = -42$.

Therefore, $S_{15} = \dfrac{56 - 42}{2} \cdot 15 = 105$.

EXAMPLE 40 If $-5 + \ldots + 49 = 616$ is the sum of the terms of a finite arithmetic sequence, how many terms are there in the sequence?

Solution Let us convert the problem into algebraic language:

$a_1 = -5$, $a_p = 49$, and $S_p = 616$, and we need to find p.

Using the sum formula,

$S_p = \dfrac{a_1 + a_p}{2} \cdot p$, that is, $616 = \dfrac{-5 + 49}{2} \cdot p$, so $p = 28$. So 28 numbers were added.

Since $a_n = a_1 + (n - 1)d$, we can also rewrite the sum formula as follows:

ALTERNATIVE SUM FORMULA

The sum of the first n terms of an arithmetic sequence is $S_n = \dfrac{2a_1 + (n-1)d}{2} \cdot n$.

EXAMPLE 41 Given an arithmetic sequence with $a_1 = -7$ and $S_{15} = -90$, find d.

Solution By using the alternative formula for the sum of first n terms, we have

$S_{15} = \dfrac{2 \cdot (-7) + (15 - 1) \cdot d}{2} \cdot 15$, that is, $-90 = \dfrac{-14 + 14d}{2} \cdot 15$, so $d = \dfrac{1}{7}$.

EXAMPLE 42 Given an arithmetic sequence with $d = 4$ and $S_9 = -189$, find a_1.

Solution Using the alternative sum formula,

$$S_n = \frac{2a_1 + (n-1)\cdot d}{2}\cdot n, \text{ and so}$$

$$S_9 = \frac{2a_1 + (9-1)\cdot 4}{2}\cdot 9$$

$$-189 = \frac{2a_1 + 32}{2}\cdot 9, \quad \text{so} \quad a_1 = -37.$$

Check Yourself 7

1. Given an arithmetic sequence with $a_1 = 4$ and $a_{10} = 15$, find S_{10}.
2. Given an arithmetic sequence with $a_{13} = 26$ and $d = -2$, find S_{13}.
3. Given an arithmetic sequence with $a_1 = 9$ and $S_8 = 121$, find d.
4. Find the sum of all the multiples of 3 between 20 and 50.

Answers

1. 95 2. 494 3. 1.75 4. 345

EXAMPLE 43 (a_n) is a sequence of consecutive integers with first term 3 and sum 52. How many terms are there in this sequence?

Solution Here $a_1 = 3$, $d = 1$, $S_n = 52$, $n = ?$.

Using the alternative sum formula,

$$S_n = \frac{2a_1 + (n-1)\cdot d}{2}\cdot n$$

$$52 = \frac{6 + (n-1)\cdot 1}{2}\cdot n$$

$$n^2 + 5n - 104 = 0.$$

Solving the quadratic equation we get $n = 8$ or $n = -13$. Since there cannot be -13 numbers, the answer is 8.

EXAMPLE 44 (a_n) is an arithmetic sequence with $S_{11} - S_{10} = 43$ and $S_{15} - S_{14} = 87$. Find d.

Solution Note that the difference between S_{11} and S_{10} is just a_{11}. Therefore, $a_{11} = 43$ and $a_{15} = 87$.

$$a_{15} = a_{11} + 4d$$
$$87 = 43 + 4d$$
$$d = 11.$$

EXAMPLE 45 (a_n) is an arithmetic sequence with $S_{12} = 30$ and $S_8 = 4$. Find a_3.

Solution Since we are looking for a term of the sequence, it is best to choose a_1 and d as our new variables.

$$\begin{cases} S_{12} = 30 \\ S_8 = 4 \end{cases}, \text{ that is } \begin{cases} \dfrac{a_1 + a_{12}}{2} \cdot 12 = 30 \\ \dfrac{a_1 + a_8}{2} \cdot 8 = 4 \end{cases}, \text{ so } \begin{cases} a_1 + a_1 + 11d = 5 \\ a_1 + a_1 + 7d = 1 \end{cases} \text{ which means } \begin{cases} a_1 = -3 \\ d = 1 \end{cases}.$$

Therefore, $a_3 = a_1 + 2d = -1$.

EXAMPLE 46 Find the general term of the arithmetic sequence (a_n) if the sum of the first n terms is $3n^2 - 4n$.

Solution $S_n = \dfrac{a_1 + a_n}{2} \cdot n$, so $3n^2 - 4n = \dfrac{a_1 + a_n}{2} \cdot n$.

Since $n \neq 0$, $3n - 4 = \dfrac{a_1 + a_n}{2}$, so $a_n = 6n - 8 - a_1$.

Choosing $n = 1$, we get $a_1 = 6 \cdot 1 - 8 - a_1$. So $a_1 = -1$.

Therefore, the general term is $a_n = 6n - 7$.

2. Applied Problems

EXAMPLE 47 The population of a city increased by 4200 in the year 2004. The rate of population growth is expected to decrease by 20 people per year. What is the city's expected total population growth between 2004 and 2014 inclusive?

Solution Note that the rate of population growth in the city is decreasing. Here, symbolically we have:

$a_1 = 4200$ (the population growth in the first year that is to be included in the total)

$d = -20$ (the difference between the population growth for consecutive years)

$S_{11} = ?$ (the total population growth in eleven years from 2004 to 2014 inclusive)

$$S_{11} = \frac{2a_1 + 10d}{2} \cdot 11 = \frac{2 \cdot 4200 + 10 \cdot (-20)}{2} \cdot 11 = 45100.$$

So the expected total population growth is 45,100 people.

EXAMPLE 48 Every hour an antique clock chimes as many times as the hour. How many times does it chime between 8:00 a.m. and 7:00 p.m. inclusive?

Solution Note that the number of chimes in the given time interval will not form an arithmetic sequence since after noon it will restart from 1. But until noon and after noon we have two independent finite arithmetic sequences. Therefore, let us define two sequences and deal with them independently.

First consider the sequence up to noon.

$a_1 = 8$ (first chime before noon)

$d = 1$ (amount of increase between consecutive chimes)

$a_p = 12$ (last chime at noon)

$S_p = ?$ (sum until noon)

$a_p = a_1 + (p-1)d$, so $12 = 8 + (p-1) \cdot 1$. So $p = 5$.

$S_p = \dfrac{a_1 + a_p}{2} \cdot p$, so $S_5 = \dfrac{8+12}{2} \cdot 5 = 50$.

Now consider the sequence after noon.

$a_1' = 1$ (first chime after noon)

$d' = 1$ (amount of increase between consecutive chimes)

$a_q' = 7$ (last chime after noon)

$S_q' = ?$ (sum after noon)

$a_q' = a_1' + (q-1)d'$, so $7 = 1 + (q-1) \cdot 1$. So $q = 7$.

$S_q' = \dfrac{a_1' + a_q'}{2} \cdot q$, so $S_7' = \dfrac{1+7}{2} \cdot 7 = 28$

Now $S_p + S_q' = ?$ (total number of chimes).

$S_p + S_q' = 50 + 28 = 78$. Therefore, the clock chimes 78 times.

Obviously, direct calculation would be a much faster way to find the correct answer in this problem, but the idea used here will be necessary in more complicated problems.

EXAMPLE 49 A farmer picks 120 tomatoes on the first day of the harvest, and each day after, he picks 40 more tomatoes than the previous day. How many days will it take for the farmer to pick a total of 3000 tomatoes?

Solution We can describe this situation with the help of arithmetic sequence notation:

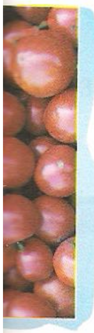
$a_1 = 120,\ d = 40,\ S_n = 3000,\ n = ?$

$$S_n = \frac{2a_1 + (n-1) \cdot d}{2} \cdot n$$

$$3000 = \frac{2 \cdot 120 + (n-1) \cdot 40}{2} \cdot n$$

$$n^2 + 5n - 150 = 0.$$

Solving the quadratic equation gives $n = -15$ or $n = 10$. Since we cannot talk about a negative number of days, the answer is ten days.



EXAMPLE 50 For a period of 42 days, each day a mailbox received four more letters than the previous day. The total number of letters received during the first 24 days of the period is equal to the total number received during the last 18 days of the period. How many letters were received during the entire period?

Solution Obviously $d = 4$ and we are looking for S_{42}. We can express the number of letters received during the first 24 days by S_{24}. But note that the number of letters received during the last 18 days of the period is not S_{18}. In fact, the number of letters received during the last 18 days is equal to the difference between the number of letters received during the entire period and the number of letters received during the first 24 days, so:

$S_{24} = S_{42} - S_{24}$ or $2 \cdot S_{24} = S_{42}$.

Using the alternative sum formula,

$$2 \cdot \frac{2a_1 + 23d}{2} \cdot 24 = \frac{2a_1 + 41d}{2} \cdot 42$$

$$(2a_1 + 92) \cdot 24 = (2a_1 + 164) \cdot 21$$

$$a_1 = 206.$$

Using the alternative sum formula once more,

$$S_{42} = \frac{2a_1 + 41d}{2} \cdot 42 = \frac{2 \cdot 206 + 41 \cdot 4}{2} \cdot 42 = 12096.$$

So during the entire period, the mailbox received 12 096 letters.

Check Yourself 8

1. Starting from 10 inclusive, is it possible to have a sum of 360 by adding a sequence of consecutive even numbers?
2. (a_n) is an arithmetic sequence with $S_{10} = 75$ and $S_6 = 9$. Find S_4.
3. Find the common difference of an arithmetic sequence if the sum of the first n terms of the sequence is given by the formula $n^2 - 2n$.
4. A free-falling object drops 9.8 meters further during each second than it did during the previous second. If an object falls 4.9 meters during the first second of its descent, how far will it fall in five seconds?

Answers

1. yes 2. –6 3. 2 4. 122.5 meters

EXAMPLE 51 $a_1, a_2, \ldots, a_{21}$ form an arithmetic sequence. The sum of the odd-numbered terms is 15 more than sum of the even-numbered terms and $a_{20} = 3a_9$. Find a_{12}.

Solution Since we are talking about two different sums, we'll divide this sequence into two different finite sequences.

Let b_n denote the odd-numbered terms with common difference $2d$, so $(b_n) = (a_1, a_3, \ldots, a_{21})$, and let S_n^b denote the sum of first n terms of this sequence. Note that for this sequence $n = 11$.

Let c_n denote the even-numbered terms with common difference $2d$, so $(c_n) = (a_2, a_4, \ldots, a_{20})$, and let S_n^c denote the sum of first n terms of this sequence. Note that for this sequence $n = 10$.

Here, note that both (b_n) and (c_n) are arithmetic sequences, and both have the same common difference which is twice the common difference of (a_n).

$$\begin{array}{ccccccccc} b_1 & & b_2 & & \cdots & b_{10} & & b_{11} \\ \uparrow & & \uparrow & & & \uparrow & & \uparrow \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_{19} & a_{20} & a_{21} \\ & \downarrow & & \downarrow & & & \downarrow & \\ & c_1 & & c_2 & \cdots & & c_{10} & \end{array}$$

Now, let us write what we are given in a system of two variables since we have two equations:

$$\begin{cases} S_{11}^b - S_{10}^c = 15 \\ a_{20} = 3a_9 \end{cases}, \text{ that is } \begin{cases} \dfrac{b_1 + b_{11}}{2} \cdot 11 - \dfrac{c_1 + c_{10}}{2} \cdot 10 = 15 \\ a_1 + 19d = 3 \cdot (a_1 + 8d) \end{cases}$$

$$\begin{cases} \dfrac{(a_1) + (a_1 + 20d)}{2} \cdot 11 - \dfrac{(a_1 + d) + (a_1 + d + 18d)}{2} \cdot 10 = 15 \\ a_1 + 19d = 3 \cdot (a_1 + 8d) \end{cases}$$

$$\begin{cases} a_1 + 10d = 15 \\ 2a_1 + 5d = 0 \end{cases}, \text{ so } \begin{cases} a_1 = -5 \\ d = 2 \end{cases}.$$

We need a_{12}, and $a_{12} = a_1 + 11d$. So $a_{12} = 17$.

EXAMPLE 52 Find the sum of all the three-digit numbers which are not divisible by 13.

Solution First of all we should realize that all the three-digit numbers which are not divisible by 13 do not form an arithmetic sequence, so we cannot use any sum formula. It will also take a long time to find and add the numbers. Therefore, let us look for a different way to express this sum.

Note that all the three-digit numbers form an arithmetic sequence, and all the three-digit numbers that *are* divisible by 13 form another arithmetic sequence, which means we can calculate these sums. Realizing that the sum we are asked to find is the difference between the sum of all three-digit numbers and the sum of all three-digit numbers that are divisible by 13, we are ready to formulize the solution.

Let S_n denote the sum of all three-digit numbers, so

$a_1 = 100,\ d = 1,\ a_n = 999,\ S_n = ?.$

$a_n = a_1 + (n-1)d$, that is $999 = 100 + n - 1$, and so $n = 900$.

$$S_n = \frac{a_1 + a_n}{2} \cdot n, \text{ so } S_{900} = \frac{100 + 999}{2} \cdot 900 = 494550.$$

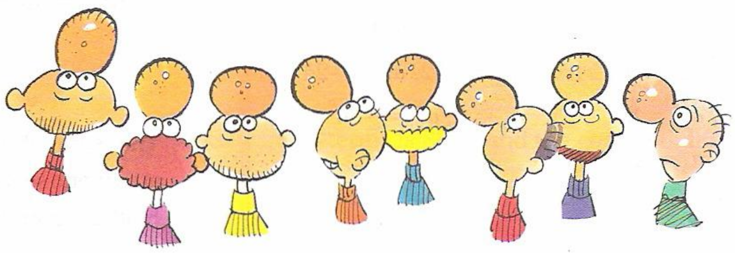
Now let S_n' denote the sum of all three-digit numbers which are divisible by 13. So,

$b_1 = 104$ (the first three-digit number that is divisible by 13), $b_n = 988$ (why?), $S_n' = ?$.

$b_n = b_1 + (n-1) \cdot d$, so $988 = 104 + (n-1) \cdot 13$ and $n = 69$.

$$S_n' = \frac{b_1 + b_n}{2} \cdot n, \text{ so } S'_{69} = \frac{104 + 988}{2} \cdot 69 = 37674.$$

We are looking for $S_{900} - S_{69}' = 494550 - 37674 = 456\ 876$. This is the sum of all the three-digit numbers which are not divisible by 13.



EXERCISES 2

A. Arithmetic Sequences

1. State whether the following sequences are arithmetic or not.

 a. $(a_n) = (n^2)$ b. $(\sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots)$ c. $(a_n) = (4n+7)$

2. Find the formula for the general term a_n of the arithmetic sequence with the given common difference and first term.

 a. $d = 2, a_1 = 3$ b. $d = \sqrt{3}, a_1 = 1$

 c. $d = 0, a_1 = 0$ d. $d = -\frac{3}{2}, a_1 = -3$

 e. $d = -1, a_1 = 0$ f. $d = 7, a_1 = \sqrt{2}$

 g. $d = b + 3, a_1 = 2b + 7$

3. Find the common difference and the general term a_n of the arithmetic sequence with the given terms.

 a. $a_1 = 3, a_2 = 5$ b. $a_1 = 4, a_4 = 10$

 c. $a_5 = \sqrt{2}, a_8 = 6\sqrt{2}$ d. $a_{12} = -12, a_{24} = -24$

 e. $a_5 = 8, a_{37} = 8$ f. $a_6 = 6, a_{20} = -34$

 g. $a_3 = 1, a_5 = 2$

 h. $a_2 = 2x - y, a_8 = x + 2y$

4. Find the general term of the arithmetic sequence using the given data.

 a. $a_{n+1} = a_n + 7, a_1 = -2$

 b. $a_{17} = 41, d = 4$

5. Fill in the blanks to form an arithmetic sequence.

 a. _, _, _, 3, _, _, _, 32.

 b. $13, \underbrace{\_, \_, \ldots, \_}_{\text{seven terms}}, 45$

6. In an arithmetic sequence the first term is –1 and the common difference is 3. Is 27 a term of this sequence?

7. Given that the following sequences are arithmetic, find the missing value.

 a. $\dfrac{a_{12} + a_{20}}{2} = ?$ b. $a_6 = \dfrac{a_4 + ?}{2}$

8. For which values of b do the following numbers form a finite arithmetic sequence?

 a. $(a_n) = (\dfrac{2}{b}, \dfrac{1}{b(1-b)}, \dfrac{2}{1-b})$

 b. $(a_n) = (5 + 2b,\ 15 + b,\ 31 - b)$

 c. $(a_n) = [(a + 1)^3, (a^3 + 3a + b), (a - 1)^3]$

9. The sum of the fifth and eighth terms of an arithmetic sequence is 24, and the tenth term is 12. Find the 20th term of the sequence.

10. Find the sum of the third and fifteenth terms of an arithmetic sequence if its ninth term is 34.

11. The sum of the third and fifth terms of an arithmetic sequence is 20, and the product of the fourth term and the sixth term is 200. Find the third term of this sequence.

B. Sum of the Terms of an Arithmetic Sequence

12. For each arithmetic sequence (a_n) find the missing value.

a. $a_1 = -5,\ a_8 = 18,\ S_8 = ?$

b. $a_1 = -3,\ a_7 = 27,\ S_{40} = ?$

c. $a_1 = 7,\ S_{16} = 332,\ d = ?$

d. $d = \frac{5}{3},\ S_{34} = 1173,\ a_1 = ?$

e. $a_1 = 2,\ a_{n+1} = a_n - 2,\ S_{23} = ?$

f. $a_1 = \frac{3}{2},\ d = \frac{1}{2},\ S_p = 1700,\ p = ?$

g. $S_{100} = 10000,\ a_{100} = 199,\ a_{10} = ?$

h. $a_n = -5n - 10,\ S_7 = ?$

i. $a_1 = 5,\ a_p = 20,\ S_p = 250,\ p = ?$

j. $S_{60} = 3840,\ a_1 = 5,\ a_{61} = ?$

k. $a_1 = 3,\ a_{10} - a_7 = -6,\ S_{20} = ?$

l. $a_1 = 1,\ S_{22} - S_{18} = 238,\ a_7 = ?$

m. $d = 5,\ S_{16} - S_{10} = 308,\ a_1 = ?$

n. $S_7 = 4 \cdot S_5,\ a_{20} = 54,\ a_1 = ?$

o. $d = 4,\ a_9 + 10 = 3a_4,\ S_{16} = ?$

p. $a_1 - d = 7,\ a_1^2 - d^2 = 91,\ S_{10} = ?$

13. Is it possible that sum of the first few terms of the arithmetic sequence (–1, 1, 3, 5, ...) is 575?

14. Given an arithmetic sequence (a_n) with $a_5 + a_8 = 27$, find S_{12}.

15. The general term of an arithmetic sequence is $a_n = 7n - 3$. Find S_{50}.

16. The sum of the first n terms of an arithmetic sequence can be formulized as $S_n = 4n^2 - 3n$. Find the first three terms of the sequence.

17. The sum of the first n terms of an arithmetic sequence can be formulized as $S_n = 2an^2$. Find d.

18. The sum of the first six terms of an arithmetic sequence is 9. The sum of the first twelve terms is 90. Find the sum of the thirteenth and seventeenth terms of this sequence.

19. The sum of the first twelve terms of an arithmetic sequence is 522. The sum of the first sixteen terms is 880. Find the common difference of this sequence.

20. In an arithmetic sequence the sum of the first six odd-numbered terms (a_1, a_3, a_5, a_7, a_9, and a_{11}) is 60. Find the sum of the first eleven terms.

21. In an arithmetic sequence the difference between the sum of the first nine terms and the sum of the first seven terms is 20. Find the sum of the first sixteen terms.

22. The sum of the squares of the fifth and eleventh terms of an arithmetic sequence is 3, and the product of the second and fourteenth terms is 1. Find the product of the first and fifteenth terms of this sequence.

23. (a_n) is an increasing arithmetic sequence such that the sum of the first three terms is 27 and the sum of their squares is 275. Find the general term of the sequence.

24. Insert 43 numbers between 3 and 25 to get an arithmetic sequence. What is the sum of all the terms?

25. A person accepts a position with a company and will receive a salary of $27,500 for the first year. The person is guaranteed a raise of $1500 per year for the first five years.

a. Determine the person's salary during the sixth year of employment.

b. Determine the total amount of money earned by the person during six full years of employment.

26. An auditorium has 30 rows of seats with 20 seats in the first row, 24 seats in the second row, 28 seats in the third row, and so on. Find the total number of seats in the anditorium.

27. A brick patio is roughly in the shape of a trapezoid. The patio has 20 rows of bricks. The first row has 14 bricks, and the twentieth row has 33 bricks. How many bricks are there in the patio?

28. A grocery worker needs to stack 30 cases of canned fruit, each containing 24 cans. He decides to display the cans by stacking them in a triangle where each row above the bottom row contains one less can. Is it possible to use all the cans and end up with a top row of only one can?

29. A runner begins running 5 km in a week. In each subsequent week, he increases the distance he runs by 1.5 km.

a. How far will he run in the twenty-second week?

b. What is the total distance the man will have covered from the beginning of the first week to the end of the twenty-second week?

30. A man climbing up a mountain climbs 800 m in the first hour and 25 m less than the previous hour in each subsequent hour. In how many hours can he climb 5700 m?

31. A well-drilling company charges $15 for drilling the first meter of a well, $15.25 for drilling the second meter, and so on. How much does it cost to drill a 100 m well?

32. Three numbers form a finite arithmetic sequence. The sum of the numbers is 3, and sum of their cubes is 4. Find the numbers.

Mixed Problems

33. The numbers a^2, b^2, and c^2 form an arithmetic sequence. Show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ also form an arithmetic sequence.

34. Solve for x.
$(x+1)+(x+4)+(x+7)+ \ldots +(x+28) = 155.$

35. Prove that if $a_{n-3} + a_{n-2} + a_{n-1} = 6n - 3$ then (a_n) is an arithmetic sequence.

36. Let (a_n) and (b_n) be two arithmetic sequences with $a_1 = 3$, $b_1 = 7$, $a_{50} + b_{50} = 190$. Find the sum of the first fifty terms of these sequences combined.

37. Two finite arithmetic sequences contain the same number of terms. The ratio of the last term of the first sequence to the first term of the second sequence is 4. The ratio of the last term of the second sequence to the first term of the first sequence is also 4. The ratio of the sum of the first sequence to the sum of second sequence is 2. Find the ratio of the common difference of the first sequence to the common difference of the second sequence.

38. (*Problem from the 18th century BC*) Divide ten slices of bread between ten people so that the second person receives 1/8 of a slice more than the first person, the third person receives 1/8 of a slice more than the second person, and so on.

39. (*Pythagoras' problem*) Find the formula for the sum of the first n odd natural numbers.

40. In an arithmetic sequence the sum of the first m terms is equal to the sum of the first n terms. Prove that the sum of first $m + n$ terms is equal to zero.

41. S_n is the sum of the first n terms of an arithmetic sequence (a_n). Show that
$S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n = 0.$

42. Find the sum of all the three-digit numbers that are not divisible by 5 or 3.

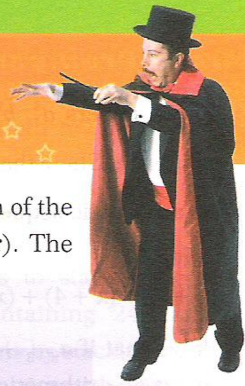
43. (a_n) is an arithmetic sequence with first terms 15, 34. (b_n) is an arithmetic sequence with first terms 7,15. Find the sum of the first thirty numbers that are common to both sequences.

44. Solve
$$\frac{x-1}{x^2} - \frac{x-2}{x^2} + \frac{x-3}{x^2} - \frac{x-4}{x^2} + \ldots - \frac{x-576}{x^2} = \frac{1}{2}.$$

45. For $p = 1, 2, \ldots, 10$ let T_p be the sum of the first forty terms of the arithmetic sequence with first term p and common difference $2p - 1$. Find $T_1 + \ldots + T_{10}$.

46. Let $ABCD$ be a trapezoid such that $AD \parallel BC$ and $AD = a$, $BC = c$. We divide non-parallel sides into $n + 1$ equal segments $n \geq 1$, by using points $M_1, M_2, \ldots, M_n \in [AB]$ and $N_1, N_2, \ldots, N_n \in [DC]$. Find $M_1N_1 + M_2N_2 + \ldots + M_nN_n$ in terms of a, c, and n.

MAGIC SQUARES

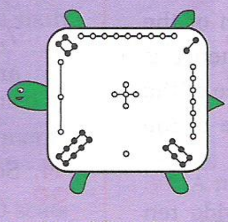


A **magic square** is an arrangement of natural numbers in a square matrix so that the sum of the numbers in each column, row, and diagonal is the same number (the **magic number**). The number of cells on one side of the square is called the **order** of the magic square.

Here is one of the earliest known magic squares:

4	9	2
3	5	7
8	1	6

It is a third order magic square constructed by using the numbers 1, 2, 3, ..., 9. Notice that the numbers in each row, column, and diagonal add up to the number 15, and 1, 2, 3, ...,9 form an arithmetic sequence. This magic square was possibly constructed in 2200 B.C. in China. It is known as the **Lo-Shu** magic square.



The famous Lo-Shu is the oldest known magic square in the world. According to the legend, the figure above was found on the back of a turtle which came from the river Lo. The word 'Shu' means 'book', so 'Lo-Shu' means 'The book of the river Lo'.

Below is another magic square, this time of order four. Note that its elements are from the finite arithmetic sequence 7, 10, 13, 16, ..., 52, and the magic number is 118.

52	13	10	43
19	34	37	28
31	22	25	40
16	49	46	7

What kind of relation exists between the sequence and the magic number? Given any finite arithmetic sequence of n^2 terms is it always possible to construct a magic square? If the numbers do not form an arithmetic sequence, is it possible to construct a magic square?

Try constructing your own magic square of order three using the numbers 4,8,12, …,36.

There are many unsolved puzzles concerning magic squares. The puzzle of Yang-Hui, which was solved in the year 2000, was one of them. According to the legend the 13th century Chinese mathematician Yang-Hui gave the emperor Sung his last magic square as a gift. This is Yang-Hui's square:

1668	198	1248
618	1038	1458
828	1878	408

+1

1669	199	1249
619	1039	1459
829	1879	409

The special property of Yang-Hui's square was that the square had elements of a finite arithmetic sequence with common difference 210 such that when 1 was added to each cell it would become another magic square with all elements prime numbers. But the emperor wanted the magic square to also give prime numbers when 1 was subtracted from each cell. He promised some land along the river to the mathematician if it was completed. Unfortunately, the life of Yang-Hui wasn't long enough to solve this puzzle. Below is the solution to the problem, calculated 725 years later:

372839669	241608569	267854789
189116129	294101009	399085889
320347229	346593449	215362349

−1

372839670	241608570	267854790
189116130	294101010	399085890
320347230	346593450	215362350

+1

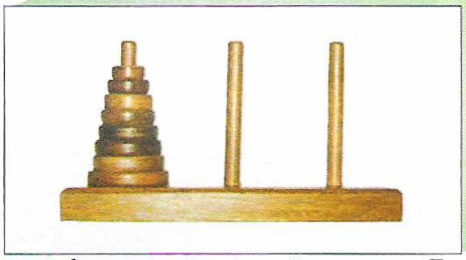
372839671	241608571	267854791
189116131	294101011	399085891
320347231	346593451	215362351

3 GEOMETRIC SEQUENCES

A. GEOMETRIC SEQUENCES

1. Definition

In the previous section, we learned about arithmetic sequences, i.e. sequences whose consecutive terms have a common difference. In this chapter we will look at another type of sequence, called a **geometric sequence**. Geometric sequences play an important role in mathematics.



A sequence is called geometric if the ratio between each consecutive term is common. For example, look at the sequence 3, 6, 12, 24, 48, …

Obviously the ratio of each term to the previous term is equal to 2, so we can formulize the sequence as $b_{n+1} = b_n \cdot 2$. The consecutive terms of the sequence have a common ratio (2), so this sequence is geometric.

For the sequence 625, 125, 25, 5, 1, … the formula will be $b_{n+1} = b_n \cdot \frac{1}{5}$. The common ratio in this sequence is $\frac{1}{5}$.

Definition

geometric sequence

If a sequence (b_n) has the same ratio q between its consecutive terms, then it is called a **geometric sequence.**

In other words, (b_n) is geometric if $b_{n+1} = b_n \cdot q$ such that $n \in \mathbb{N}$, $q \in \mathbb{R}$. q is called the **common ratio** of the sequence. In this book, from now on we will use b_n to denote the general term of a geometric sequence, and q to denote the common ratio.

If $q > 1$, the geometric sequence is **increasing** when $b_1 > 0$ and **decreasing** when $b_1 < 0$.

If $0 < q < 1$, geometric sequence is **increasing** when $b_1 < 0$ and **decreasing** when $b_1 > 0$.

If $q < 0$, then the sequence is **not monotone**.

What can you say if $q = 1$? What about $q = 0$?

EXAMPLE 53 State whether the following sequences are geometric or not. If a sequence is geometric, find the common ratio.

a. 1, 2, 4, 8, … b. 3, 3, 3, 3, … c. 1, 4, 9, 16, … d. $5, -1, \frac{1}{5}, -\frac{1}{25}, \dots$

Solution a. geometric, $q = 2$ b. geometric, $q = 1$ c. not geometric d. geometric, $q = -\frac{1}{5}$

EXAMPLE 54 State whether the sequences with the given general terms are geometric or not. If a sequence is geometric, find the common ratio.

a. $b_n = 3^n$ b. $b_n = n^2 + 3$ c. $b_n = 3 \cdot 2^{n+3}$ d. $b_n = 3n + 5$

Solution a. $b_{n+1} = 3^{n+1}$, so the ratio between each consecutive term is $\frac{b_{n+1}}{b_n} = \frac{3^{n+1}}{3^n} = 3$, which is constant. So (b_n) is a geometric sequence and $q = 3$.

b. $b_{n+1} = (n + 1)^2 + 3$, so the ratio between each consecutive term is

$\frac{b_{n+1}}{b_n} = \frac{(n+1)^2+3}{n^2+3} = \frac{n^2+2n+4}{n^2+3}$, which is not constant. So (b_n) is not a geometric sequence.

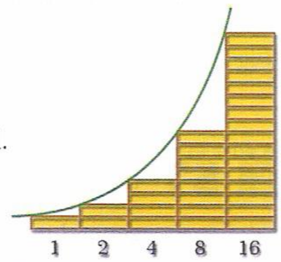
c. $b_{n+1} = 3 \cdot 2^{n+4}$, so the ratio between each consecutive term is $\frac{b_{n+1}}{b_n} = \frac{3 \cdot 2^{n+4}}{3 \cdot 2^{n+3}} = 2$, which is constant. So (b_n) is a geometric sequence and $q = 2$.

d. Since the general term has a linear form, this is an arithmetic sequence. It is not geometric.

With the help of the above example we can see that if the formula for the general term of a sequence gives us an exponential function with a linear exponent (a function with only one exponent variable), then it is geometric.

Note

The general term of a geometric sequence is exponential.



Geometric growth is exponential.

2. General Term

We have seen that for a geometric sequence, $b_{n+1} = b_n \cdot q$. This formula is defined recursively. If we want to make faster calculations, we need to express the general term of a geometric sequence more directly. The formula is derived as follows:

If (b_n) is geometric, then we only know that $b_{n+1} = b_n \cdot q$. Let us write a few terms.

$$b_1 = b_1$$
$$b_2 = b_1 \cdot q$$
$$b_3 = b_2 \cdot q = (b_1 \cdot q) \cdot q = b_1 \cdot q^2$$
$$b_4 = b_3 \cdot q = (b_1 \cdot q^2) \cdot q = b_1 \cdot q^3$$
$$b_5 = b_1 \cdot q^4$$
$$\vdots$$
$$b_n = b_1 \cdot q^{n-1}$$

This is the general term of a geometric sequence.

GENERAL TERM FORMULA

The general term of a geometric sequence (b_n) with common ratio q is

$$b_n = b_1 \cdot q^{n-1}$$

EXAMPLE 55 If 100, 50, 25 are the first three terms of a geometric sequence (b_n), find the sixth term.

Solution We can calculate the common ratio as $q = \frac{b_3}{b_2} = \frac{b_2}{b_1} = \frac{1}{2}$, so $b_1 = 100$, $q = \frac{1}{2}$.

Using the general term formula, $b_n = b_1 \cdot q^{n-1}$, so $b_6 = 100 \cdot (\frac{1}{2})^{6-1} = \frac{25}{8}$.

EXAMPLE 56 (b_n) is a geometric sequence with $b_1 = \frac{1}{3}$, $q = 3$. Find b_4.

Solution Using the general term formula,

$b_n = b_1 \cdot q^{n-1}$. Therefore, $b_4 = \frac{1}{3} \cdot 3^{4-1} = 9$.

How can you relate this building to a geometric sequence?

EXAMPLE 57 (b_n) is a geometric sequence with $b_1 = -15$, $q = \frac{1}{5}$. Find the general term.

Solution Using the general term formula, $b_n = b_1 \cdot q^{n-1}$.

Therefore, $b_n = -15 \cdot \left(\frac{1}{5}\right)^{n-1} = -15 \cdot \left(\frac{1}{5}\right)^{n} \cdot \left(\frac{1}{5}\right)^{-1} = -75 \cdot \left(\frac{1}{5}\right)^{n}$.

EXAMPLE 58 Consider the geometric sequence (b_n) with $b_1 = \frac{1}{9}$ and $q = 3$. Is 243 a term of this sequence?

Solution Using the general term formula,

$b_n = b_1 \cdot q^{n-1}$ and so $b_n = \frac{1}{9} \cdot 3^{n-1}$.

Now $243 = \frac{1}{9} \cdot \frac{3^n}{3}$, and so $3^n = 3^8$. Therefore, $n = 8$.

Since 8 is a natural number, 243 is the eighth term of this sequence.

038

EXAMPLE 59 In a monotone geometric sequence $b_1 \cdot b_5 = 12$, $\frac{b_2}{b_4} = 3$. Find b_2.

Solution $\frac{b_2}{b_4} = 3$, that is $\frac{b_1 \cdot q}{b_1 \cdot q^3} = 3$. So $q = \pm\frac{1}{\sqrt{3}}$.

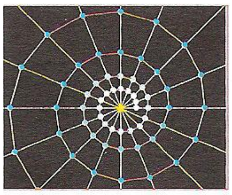
Since the sequence is monotone, we take $q = \frac{1}{\sqrt{3}}$.

$b_1 \cdot b_5 = 12$, that is $b_1 \cdot b_1 \cdot q^4 = 12$.

$b_1^2 \cdot \frac{1}{9} = 12$, that is $b_1 = 6\sqrt{3}$. So $b_2 = b_1 \cdot q = 6\sqrt{3} \cdot \frac{1}{\sqrt{3}} = 6$.

Why? Would the answer change if the sequence was not monotone? Why?

Check Yourself 9



1. Is the sequence with general term $b_n = \frac{1}{3} \cdot 4^{n+3}$ a geometric sequence? Why?
2. $\frac{3}{16}, \frac{3}{8}, \frac{3}{4}$ are the first three terms of a geometric sequence (b_n). Find the eighth term.
3. (b_n) is a non-monotone geometric sequence with $b_1 = \frac{1}{4}$, $b_7 = 16$. Find the common ratio of the sequence and b_4.
4. (b_n) with is a geometric sequence with $b_1 = -3$, $q = -2$. Is -96 a term of this sequence?

Answers

1. yes, because the general term formula is exponential 2. 24 3. $q = -2$; $b_4 = -2$ 4. no

3. Advanced General Term Formula

EXAMPLE 60 (b_n) is a geometric sequence with $b_4 = 56$, $q = -\frac{1}{2}$. Find b_9.

Solution $b_4 = b_1 \cdot q^3$, that is $56 = b_1 \cdot (-\frac{1}{2})^3$. So $b_1 = -448$.

$b_9 = b_1 \cdot q^8 = -448 \cdot (-\frac{1}{2})^8 = -\frac{7}{4}$

In this example, we calculated the first term of the sequence (b_1) from b_4, then used this value to find b_9. However, there is a quicker way to solve this problem: in general, if we know the common ratio and any term of a geometric sequence, we can find the required term without finding the first term. Look at the calculation:

If we know b_p and q, to express b_n we can write:

$$b_n = b_1 \cdot q^{n-1} \qquad (1)$$

$$b_p = b_1 \cdot q^{p-1}. \qquad (2)$$

Making a side-by-side division of (1) by (2), we get $\dfrac{b_n}{b_p} = q^{n-p}$.

So $b_n = b_p \cdot q^{n-p}$.

ADVANCED GENERAL TERM FORMULA

The general term of a geometric sequence (b_n) with common ratio q is $\boldsymbol{b_n = b_p \cdot q^{n-p}}$, where b_p is any term of the sequence.

So using the advanced general term formula, we can solve the previous example as follows:

$$b_n = b_p \cdot q^{n-p}$$

$$b_9 = b_4 \cdot q^5 = 56 \cdot (-\frac{1}{2})^5 = -\frac{7}{4}.$$

Here, it is not important which term you write in the place of b_n and b_p.

Note that when $p = 1$, the advanced general term formula becomes the general term formula we studied previously.

EXAMPLE 61 (b_n) is a geometric sequence with $b_5 = \dfrac{1}{32}$, $b_8 = 4^{-4}$. Find the common ratio.

Solution We have $b_5 = \dfrac{1}{32} = 2^{-5}$ and $b_8 = 4^{-4} = 2^{-8}$.

Using the advanced general term formula,

$$b_n = b_p \cdot q^{n-p}$$

$$b_8 = b_5 \cdot q^3$$

$$2^{-8} = 2^{-5} \cdot q^3, \text{ so } q = \sqrt[3]{\frac{2^{-8}}{2^{-5}}} = \frac{1}{2}.$$

4. Common Ratio Formula

Let us formulize the procedure in the last example, which helps us to find the common ratio of a geometric sequence with any two terms b_p and b_r such that $p > r$.

Applying the advanced general term formula, $b_p = b_r \cdot p^{p-r}$, so $\dfrac{b_p}{b_r} = q^{p-r}$.

If $p - r$ is even, $q = \pm\sqrt[p-r]{\dfrac{b_p}{b_r}}$.

If $p - r$ is odd, $q = \sqrt[p-r]{\dfrac{b_p}{b_r}}$.

(Why did we define $p > r$?)

COMMON RATIO FORMULA

The common ratio of a geometric sequence (b_n) with terms b_p and b_r is

$$q = \begin{cases} \pm\sqrt[p-r]{\dfrac{b_p}{b_r}}, \text{if } p-r \text{ is even} \\ \sqrt[p-r]{\dfrac{b_p}{b_r}}, \text{if } p-r \text{ is odd} \end{cases} \quad \text{where } p > r.$$

EXAMPLE 62 Given a monotone geometric sequence (b_n) with $b_3 = 9$, $b_5 = 16$, find the common ratio.

Solution Using the common ratio formula,

$q = \pm\sqrt[5-3]{\dfrac{b_5}{b_3}} = \pm\dfrac{4}{3}$. Since the sequence is monotone, $q = \dfrac{4}{3}$. Otherwise, one term would be negative and the next would be positive, and that would give a sequence which is neither increasing nor decreasing. Note that if we did not know that the sequence was monotone, then there would be two possible answers.

EXAMPLE 63 (b_n) is a non-monotone geometric sequence with $b_2 = 2, b_4 = \dfrac{8}{9}$. Which term is $\dfrac{32}{81}$?

Solution Since the sequence is not monotone, the common ratio is negative. Using the common ratio formula, $q = -\sqrt[4-2]{\dfrac{b_4}{b_2}} = -\sqrt{\dfrac{b_4}{b_2}} = -\dfrac{2}{3}$. If $\dfrac{32}{81}$ is a term, then

$b_p = b_2 \cdot q^{p-2}$, that is $\dfrac{32}{81} = 2 \cdot (-\dfrac{2}{3})^{p-2}$, so $(\dfrac{2}{3})^4 = (-\dfrac{2}{3})^{p-2}$, which means $p = 6$.

Since 6 is a natural number, $\dfrac{32}{81}$ is the sixth term.

5. Middle Term Formula (Geometric Mean)

EXAMPLE 64 Given a geometric sequence (b_n) with $b_8 = 10$, find $b_2 \cdot b_{14}$.

Solution This time we have just one value as data. Since the formulas we have learned up to now depend on more than one data value, it is impossible to find b_2 or b_{14}. However, we are not asked to find b_2 or b_{14}, but to find $b_2 \cdot b_{14}$.

Let us apply the advanced general term formula, keeping in mind that we just know b_8:

$$b_2 = b_8 \cdot q^{2-8} \qquad (1)$$
$$b_{14} = b_8 \cdot q^{14-8}. \qquad (2)$$

Multiplying (1) by (2), we get

$b_2 \cdot b_{14} = b_8^2 = 10^2 = 100.$

The solution to the previous example gives us a practical formula.

Let b_p and b_k be two terms of a geometric sequence such that $k < p$. Then,

$$b_{p-k} = b_p \cdot q^{-k} \qquad (1)$$
$$b_{p+k} = b_p \cdot q^{k}. \qquad (2)$$

Multiplying (1) and (2) we get

$b_{p-k} \cdot b_{p+k} = b_p^{\ 2}$ or $b_p = \pm\sqrt{b_{p-k} \cdot b_{p+k}}$, which means that the square of any term x in a geometric sequence is equal to the product of any two terms that are at equal distance from x in the sequence. In the previous example note that b_8 was at equal distance from of b_2 and b_{14}. (Could we solve the problem if we were given b_8 instead of b_{10}?)

MIDDLE TERM FORMULA (Geometric Mean)

In a geometric sequence $b_p^2 = b_{p-k} \cdot b_{p+k}$ where $k < p$.

The geometric mean of two numbers x and y is m if $m = \sqrt{xy}$.

Note that m is the same distance from x as from y, so x, m, y form a finite geometric sequence.

For example, all the following equalities will hold in a geometric sequence.

$b_2^{\ 2} = b_1 \cdot b_3$ since b_2 is in the middle of b_1 and b_3.

$b_7^{\ 2} = b_5 \cdot b_9 = b_1 \cdot b_{13} = b_2 \cdot b_x$ (x must be 12)

$b_{10} \cdot b_{20} = b_y^{\ 2}$ (y must be 15)

EXAMPLE 65 1, x, 9 are three consecutive terms of a geometric sequence. Find x.

Solution If we say $b_1 = 1$, $b_2 = x$, $b_3 = 9$, then using the middle term formula, $b_2^2 = b_1 \cdot b_3$, i.e. $x^2 = 1 \cdot 9$. Therefore, x is 3 or –3 if the sequence is geometric.

Note

Three numbers a, b, c form consecutive terms of a geometric sequence if and only if $b^2 = a \cdot c$.

EXAMPLE 66 Find the common ratio q for the geometric sequence (b_n) with $b_1 = 32$ and $b_2 \cdot b_9 = 2$.

Solution Using the middle term formula, we get $b_2 \cdot b_9 = b_{5.5}^2$, which is nonsense!

Realizing that we are given b_1, let's write another nonsense equation: $b_1 \cdot b_{10} = b_{5.5}^2$.

We know that there is no $b_{5.5}$ but we have $b_2 \cdot b_9$ and $b_1 \cdot b_{10}$ which are equal. That is,

$b_1 \cdot b_{10} = b_2 \cdot b_9$, so $32 \cdot b_{10} = 2$. Therefore, $b_{10} = \frac{1}{16}$.

Now using the general term formula,

$b_{10} = b_1 \cdot q^9$, so $\frac{1}{16} = 32 \cdot q^9$. Therefore, $q = \frac{1}{2}$.

Check Yourself 10

1. (b_n) is a geometric sequence with $b_4 = 12$ and $q = \frac{1}{3}$. Find b_7.
2. (b_n) is a geometric sequence with $b_7 = 9$ and $b_{10} = 72$. Find the common ratio.
3. (b_n) is a geometric sequence with $b_5 = \frac{5}{4}$ and $b_8 = 10$. Find b_{10}.
4. Fill in the blanks if the following numbers form a geometric sequence: –2, __, __, __, –162.

Answers

1. $\frac{4}{9}$ 2. 2 3. 40 4. –6, –18, –54 or 6, –18, 54

EXAMPLE 67 Given a monotone geometric sequence (b_n) with $b_1 + b_5 = 30$, $b_3 + b_7 = 120$, find b_1.

Solution We must express these two equations in terms of two variables, say b_1 and q.

$$\begin{cases} b_1 + b_5 = 30 \\ b_3 + b_7 = 120 \end{cases}, \text{ so } \begin{cases} b_1 + b_1 \cdot q^4 = 30 \\ b_1 \cdot q^2 + b_1 \cdot q^6 = 120 \end{cases}, \text{ so } \begin{cases} b_1 \cdot (1 + q^4) = 30 & (1) \\ b_1 \cdot q^2 \cdot (1 + q^4) = 120 & (2) \end{cases}.$$

Dividing equation (2) by equation (1), we get

$q^2 = 4$, so $q = \pm 2$.

Since the sequence is monotone, $q = 2$.

Using equation (1): $b_1 \cdot (1 + 2^4) = 30$, so $b_1 = \frac{30}{17}$.

EXAMPLE 68 Three numbers form a geometric sequence. If we increase the second number by 2, we get an arithmetic sequence. After this, if we increase the third number by 9, we get a geometric sequence again. Find the three initial numbers.

Solution Since we are given three numbers, let us solve this problem with the help of the middle term formulas for arithmetic and geometric sequences. Naming these numbers a, b, and c respectively, we have:

a, b, c	geometric sequence
$a, b+2, c$	arithmetic sequence
$a, b+2, c+9$	geometric sequence

So we have the following system of three equations with three unknowns:

$$\begin{cases} b^2 = ac \\ b+2 = \dfrac{a+c}{2} \\ (b+2)^2 = a(c+9) \end{cases}, \text{ that is } \begin{cases} b^2 = ac & (1) \\ 2b+4 = a+c & (2) \\ b^2+4b+4 = \underbrace{ac}_{b^2}+9a & (3) \end{cases}$$

Using (3) in (1), $b^2 = \dfrac{4b+4}{9} \cdot c$, so $c = \dfrac{9b^2}{4b+4}$. (4)

Using (3) and (4) in (2), $2b + 4 = \dfrac{4b+4}{9} + \dfrac{9b^2}{4b+4}$, so $25b^2 - 184b - 128 = 0$.

Solving the quadratic equation, we get $b = -\dfrac{16}{25}$ or $b = 8$. Substituting these numbers in (3) and (4) we find a and c respectively.

So the system will have $\begin{cases} a = \dfrac{4}{25} \\ b = -\dfrac{16}{25} \\ c = \dfrac{64}{25} \end{cases}$ or $\begin{cases} a = 4 \\ b = 8 \\ c = 16 \end{cases}$ as possible solution sets.

Geometric growth is exponential!

EXAMPLE 69 Find four numbers forming a geometric sequence such that the second term is 35 less than the first term and the third term is 560 more than the fourth term.

Solution For convenience, let us denote the terms by a, b, c, d, and the common ratio as usual by q. Our data now looks like the following:

$$\begin{cases} b = a - 35 \\ c = d + 560. \end{cases}$$

We have to reduce the number of variables to two using the fact that we have a geometric sequence.

$$\begin{cases} aq = a - 35 \\ aq^2 = aq^3 + 560 \end{cases}, \text{ so } \begin{cases} a = \dfrac{35}{1-q} & (1) \\ \dfrac{35}{1-q} \cdot q^2 = \dfrac{35}{1-q} \cdot q^3 + 560 & (2) \end{cases}.$$

Solving equation (2), we get $q = \pm 4$.

If $q = -4$, then $a = 7$, $b = -28$, $c = 112$, $d = -448$.

If $q = 4$, then $a = -\dfrac{35}{3}$, $b = -\dfrac{140}{3}$, $c = -\dfrac{560}{3}$, $d = -\dfrac{2240}{3}$.

Both of these sets of values are possible solution sets for the problem.

B. SUM OF THE TERMS OF A GEOMETRIC SEQUENCE

1. Sum of the First n Terms

Let us consider the geometric sequence with first few terms 1, 2, 4, 8, 16.

The sum of the first term of this sequence is obviously 1. The sum of the first two terms is 3, the sum of the first three terms is 7, and so on. To write this in a more formal way, let us use S_n to denote **the sum of the first n terms**, i.e. $S_n = b_1 + b_2 + \ldots + b_n$. Now,

$$S_1 = 1$$
$$S_2 = 1 + 2 = 3$$
$$S_3 = 1 + 2 + 4 = 7$$
$$S_4 = 1 + 2 + 4 + 8 = 15$$
$$S_5 = 1 + 2 + 4 + 8 + 16 = 31.$$

EXAMPLE 70 Given the geometric sequence with general term $b_n = 3 \cdot (-2)^n$, find the sum of first three terms.

Solution $S_3 = b_1 + b_2 + b_3 = -6 + 12 - 24 = -18$

How could we find S_{100} in the previous example? Calculating terms and finding their sums takes time and effort for large sums. As geometric sequences grow very fast, we need a more efficient way of calculating these sums. The following theorem meets our needs:

Theorem

The sum of the first n terms of a geometric sequence (b_n) is $S_n = b_1 \cdot \dfrac{1-q^n}{1-q}$, $q \neq 1$.

Proof

$S_n = b_1 + b_2 + b_3 + \ldots + b_{n-1} + b_n$

$S_n = b_1 + b_1 \cdot q + b_1 \cdot q^2 + \ldots + b_1 \cdot q^{n-2} + b_1 \cdot q^{n-1} \qquad (1)$

$q \cdot S_n = b_1 \cdot q + b_1 \cdot q^2 + b_1 \cdot q^3 + \ldots + b_1 \cdot q^{n-1} + b_1 \cdot q^n \qquad (2)$

Subtracting (2) from (1), we get

$S_n - q \cdot S_n = b_1 - b_1 \cdot q^n$

$S_n = b_1 \cdot \dfrac{1-q^n}{1-q}$

EXAMPLE 71 Given a geometric sequence with $b_1 = \dfrac{1}{81}$ and $q = 3$, find S_6.

Solution Using the sum formula,

$S_n = b_1 \cdot \dfrac{1-q^n}{1-q}$, so $S_6 = \dfrac{1}{81} \cdot \dfrac{1-3^6}{1-3} = \dfrac{364}{81}$.

EXAMPLE 72 Given a geometric sequence with $S_6 = 3640$ and $q = 3$, find b_1.

Solution Using the sum formula,

$S_6 = b_1 \cdot \dfrac{1-q^6}{1-q}$, so $3640 = b_1 \cdot \dfrac{1-3^6}{1-3}$, and so $b_1 = 10$.

EXAMPLE 73 Given a geometric sequence with $q = \dfrac{1}{3}$, $b_p = 5$ and $S_p = 1820$, find b_1.

Solution Using the sum formula,

$S_p = b_1 \cdot \dfrac{1-q^p}{1-q} = \dfrac{b_1 - b_1 \cdot q^p}{1-q} = \dfrac{b_1 - b_{p+1}}{1-q} = \dfrac{b_1 - b_p \cdot q}{1-q}$, so $1820 = \dfrac{b_1 - 5 \cdot \frac{1}{3}}{1 - \frac{1}{3}}$. Therefore, $b_1 = 1215$.

EXAMPLE 74 Given a geometric sequence with $b_1 = 3$ and $S_3 = \frac{19}{3}$, find q.

Solution Using the sum formula,

$x^3 - y^3 = (x-y)(x^2+xy+y^2)$
$x^3 + y^3 = (x+y)(x^2-xy+y^2)$

$S_3 = b_1 \cdot \frac{1-q^3}{1-q}$, and so $\frac{19}{3} = 3 \cdot \frac{(1-q)(1+q+q^2)}{1-q}$. Therefore, $\frac{19}{9} = 1+q+q^2$.

Solving the quadratic equation, we get $q = -\frac{5}{3}$ or $q = \frac{2}{3}$.

Check Yourself 11

1. Given a geometric sequence with $b_1 = 1$ and $q = -2$, find S_7.
2. Given a geometric sequence with $S_9 = 513$ and $q = -2$, find b_5.
3. Given a geometric sequence with $q = 2$, $b_1 = 7$, and $S_p = 896$, find p.
4. Given a geometric sequence with $b_1 = 192$ and $S_3 = 252$, find q.

Answers

1. 43 2. 48 3. 8 4. $-\frac{5}{4}$ or $\frac{1}{4}$

EXAMPLE 75 Given a monotone geometric sequence with $b_4 - b_2 = -\frac{45}{32}$, $b_6 - b_4 = -\frac{45}{512}$, find b_1 and q.

Solution Let us write the given equations in terms of b_1 and q.

$$\begin{cases} b_1 \cdot q^3 - b_1 \cdot q = -\dfrac{45}{32} \\ b_1 \cdot q^5 - b_1 \cdot q^3 = -\dfrac{45}{512} \end{cases}, \text{ so } \begin{cases} b_1 \cdot q \cdot (q^2 - 1) = -\dfrac{45}{32} & (1) \\ b_1 \cdot q^3 \cdot (q^2 - 1) = -\dfrac{45}{512} & (2) \end{cases}.$$

Dividing (2) by (1), we get $q^2 = \frac{1}{16}$, so $q = \pm\frac{1}{4}$.

Since the sequence is monotone, we take $q = \frac{1}{4}$.

Using this information in equation (1) we get $b_1 = 6$.

EXAMPLE 76 Given a geometric sequence with $S_7 = 14$ and $S_{14} = 18$, find $b_{15} + \ldots + b_{21}$.

Solution Clearly, $b_{15} + \ldots + b_{21} = S_{21} - S_{14}$.

However, we are given S_7 and S_{14}, so we need to find a way of expressing S_{21} in terms of the given data.

$$S_{21} = b_1 \cdot \frac{1-q^{21}}{1-q} = b_1 \cdot \frac{(1-q^7)(1+q^7+q^{14})}{1-q} \qquad (1)$$

$$S_{14} = b_1 \cdot \frac{1-q^{14}}{1-q} = b_1 \cdot \frac{(1-q^7)(1+q^7)}{1-q} \qquad (2)$$

$$S_7 = b_1 \cdot \frac{1-q^7}{1-q} \qquad (3)$$

Dividing (1) by (3) we get,

$$\frac{S_{21}}{S_7} = 1+q^7+q^{14}. \qquad (4)$$

Dividing (2) by (3) we get,

$$\frac{S_{14}}{S_7} = 1+q^7, \qquad (5)$$

so $q^7 = \frac{S_{14}}{S_7} - 1 = \frac{18}{14} - 1 = \frac{2}{7}$.

Subtracting (5) from (4) we get,

$$\frac{S_{21}-S_{14}}{S_7} = q^{14}, \text{ so } S_{21} - S_{14} = (q^7)^2 \cdot S_7 = \frac{4}{49} \cdot 14 = \frac{8}{7}.$$

EXAMPLE 77 (b_n) is a geometric sequence such that the sum of the first three terms is 91, and the terms $b_1 + 25$, $b_2 + 27$, $b_3 + 1$ form an arithmetic sequence. Find b_1.

Solution Using the sum formula,

$$S_3 = b_1 \cdot \frac{1-q^3}{1-q} = b_1 \cdot \frac{(1-q)(1+q+q^2)}{(1-q)}, \text{ so } b_1 \cdot (1+q+q^2) = 91. \qquad (1)$$

Using the middle term formula for arithmetic sequences,

$$b_2 + 27 = \frac{b_1 + 25 + b_3 + 1}{2}, \quad \text{so} \quad \underbrace{b_1 \cdot q}_{\substack{\text{since } b_2 \text{ is} \\ \text{a term of a} \\ \text{geometric sequence}}} + 27 = \frac{b_1 + 25 + b_1 \cdot q^2 + 1}{2}.$$

Now we have, $b_1 \cdot (q^2 - 2q + 1) = 28$. (2)

Dividing (1) by (2) we get,

$$\frac{1+q+q^2}{q^2-2q+1}=\frac{13}{4}, \text{ so } 3q^2-10q+3=0.$$

This quadratic equation gives two solutions: $q=\frac{1}{3}$ or $q = 3$.

If $q=\frac{1}{3}$, then using equation (1) or (2) we get $b_1 = 63$.

If $q = 3$, then using equation (1) or (2) we get $b_1 = 7$.

Both of these are possible values for b_1.

2. Applied Problems

EXAMPLE 78 After the accelerator pedal of a car is released, the driver of the car waits five seconds before applying the brakes. During each second after the first, the car covers 0.9 times the distance it covered during the preceding second. If the car moved 20 m during the first second, how far does it move before the brakes are applied?

Solution Here we have,

$b_1 = 20$ (distance covered in the first second)

$q = 0.9$ (the ratio of distance covered to the distance covered in the preceding second)

$S_5 = ?$ (total distance covered in five seconds)

Using the sum formula,

$$S_5 = b_1 \cdot \frac{1-q^5}{1-q} = 20 \cdot \frac{1-0.9^5}{1-0.9} = 81.902.$$

Therefore, before the brakes are applied the car moves 81.902 m.

EXAMPLE 79 How many ancestors from parents through great-great-great grandparents do three unrelated people have?

Solution Let's try to formulize the problem. Each person has two parents, a mother and a father, and these people are distinct because the people in the problem are unrelated. These parents are the closest generation to the original people; we can call them the first generation. Now, each person in the first generation also has two different parents, which we can call the second generation. If we continue like this, we can see that there are five generations, and each generation contains twice the number of people of the previous generation. This is a geometric sequence, and we can write,

$b_1 = 6$ (total number of parents of the three unrelated people)

$q = 2$ (the ratio between the number of people in successive generations)

$S_5 = ?$ (the total number of ancestors in five generations).

Using the sum formula,

$$S_5 = 6 \cdot \frac{1-2^5}{1-2} = 186.$$

So the three unrelated people will have 186 ancestors from parents through great-great-great grandparents.

EXAMPLE 80 A set of five weights has a total mass of 930 g. If the weights are arranged in order from the lightest to the heaviest, the second weight has twice the mass of the first, and so on. What is the mass of the heaviest weight?

Solution Let us formulize the problem:

$S_5 = 930,\ q = 2,\ b_5 = ?.$

Using the sum formula,

$$S_5 = b_1 \cdot \frac{1-q^5}{1-q}, \text{ then } 930 = b_1 \cdot \frac{1-2^5}{1-2}, \text{ so } b_1 = 30.$$

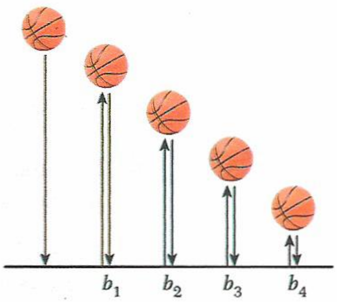
Using the general term formula, $b_5 = b_1 \cdot q^4 = 30 \cdot 2^4 = 480$.

Therefore, the heaviest weight has a mass of 480 g.

EXAMPLE 81 A ball is dropped from a height of 81 cm. Each time it bounces, it returns to $\frac{2}{3}$ of its previous height. What is the total distance the ball has traveled in the air when it hits the ground for the fifth time?

Solution Choosing $b_1 = 81,\ q = \frac{2}{3},\ S_5 = ?$ won't give us the answer that is required. To understand why, let us look at the distance that the ball travels using the diagram opposite.

We can see that except the first 81 cm, each length is covered twice. So if we define a geometric sequence which has $81 \cdot \frac{2}{3}$ as the first term, we can formulize our answer as,



$$\text{Total distance} = \underbrace{81}_{\text{first fall}} + \left(\underbrace{2}_{\text{rise and fall}} \cdot \underbrace{\underbrace{81 \cdot \frac{2}{3}}_{b_1 \text{ in the figure}} \cdot \frac{1-\left(\frac{2}{3}\right)^4}{1-\frac{2}{3}}}_{\text{sum formula for the first four terms}} \right) = 341 \text{ cm}.$$

Check Yourself 12

1. Given a monotone geometric sequence with $b_4 - b_2 = -\frac{45}{32}$ and $b_6 - b_4 = -\frac{45}{512}$, find b_1 and q.
2. A tree loses 384 leaves during the first week of fall and $\frac{3}{2}$ as many leaves in each successive week. At the end of seven weeks all the leaves have fallen. How many leaves did the tree have at the start of fall?

Answers

1. $b_1 = 6,\ q = \frac{1}{4}$ 2. 12 354 leaves

C. INFINITE SUM OF A GEOMETRIC SEQUENCE (OPTIONAL)

1. Infinite Sum Formula

In geometric sequences with common ratio between –1 and –1, each successive term in the sequence gets closer to zero. We can easily see this in the following examples:

when $q = \frac{1}{2}$, $(b_n) = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots)$,

when $q = -\frac{1}{30}$, $(b_n) = (3, -\frac{1}{10}, \frac{1}{300}, -\frac{1}{9000}, \ldots)$.

In both examples, the terms get closer to zero as n increases. In the second example the approach is more rapid than in the first, and the sequence alternates between positive and negative numbers.

A simple investigation with a few more examples will quickly reveal that for geometric sequences with common ratio $-1 < q < 1$, as n increases the total sum of the terms (S_n) eventually settles down to a constant value. In other words, we can find the **infinite sum** of a geometric sequence with common ratio $-1 < q < 1$.

EXAMPLE 82 Find $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

Solution Clearly each term of this sum is a term of the geometric sequence with $b_1 = 1$ and $q = \frac{1}{2}$. We are looking for the infinite sum, i.e. S_∞.

Using the sum formula,

$$S_\infty = b_1 \cdot \frac{1-q^\infty}{1-q} = 1 \cdot \frac{1-\left(\frac{1}{2}\right)^\infty}{1-\frac{1}{2}} = 1 \cdot \frac{1-0}{\frac{1}{2}} = 2.$$

Here, $(\frac{1}{2})^\infty = \frac{1^\infty}{2^\infty} = \frac{1}{2^\infty} = 0,$ since 1 has no significance next to 2^∞.

We now have an equation which helps us to calculate the infinite sum of a geometric sequence.

Theorem

The infinite sum of a geometric sequence (b_n) with common ratio $|q| < 1$ is denoted by S, and is given by the formula $S = \dfrac{b_1}{1-q}$.

Proof $S_n = b_1 \cdot \dfrac{1-q^n}{1-q}$ by the general sum formula. If we choose $n \to \infty$, $S = b_1 \cdot \dfrac{1-\overbrace{0}^{\text{since } |q|<1}}{1-q} = \dfrac{b_1}{1-q}$.

Note

Remember that the total sum of terms only settles at a constant value if $-1 < q < 1$.
If $|q| \geq 1$, then the geometric sequence has no infinite sum.

EXAMPLE 83 Find 100 + 50 + 25 + ...

Solution Here $b_1 = 100$ and $q = \frac{1}{2}$. Using the infinite sum formula, $S = \dfrac{100}{1-\frac{1}{2}} = 200.$

EXAMPLE 84 Find –5 + 10 – 20 + ...

Solution Here, $q = -2$. Therefore, there is no infinite sum. $(-2 < -1)$.

2. Repeating Decimals

When we use a calculator, at the end of division we often have rational numbers with repeating decimals, i.e. decimals with a repeating sequence of one or more digits in the fraction part. We can use our knowledge of the infinite sum of a geometric sequence to write repeating decimals as fractions.

Note

We can write a repeating decimal such as 0.66666... as $0.\overline{6}$ or 0.(6). In this book, we use the first notation.

EXAMPLE 85 Write the number $0.\overline{72}$ as a fraction.

Solution Let us try to see the geometric sequence in this question.

$$\begin{aligned} 0.\overline{72} &= 0.727272... \\ &= 0.72 + 0.0072 + 0.000072 + ... \\ &= 0.72 + 0.72 \cdot 0.01 + 0.72 \cdot 0.0001 + ... \\ &= 0.72 + 0.72 \cdot 0.01 + 0.72 \cdot (0.01)^2 + ... \end{aligned}$$

Now we can see that each term of this sum is a term of the geometric sequence with $b_1 = 0.72$, $q = 0.01$ and we are looking for the infinite sum, that is S.

Using the infinite sum formula,

$$S = \frac{0.72}{1-0.01} = \frac{72}{99} = \frac{8}{11}.$$

EXAMPLE 86 Write the number $2.1\overline{5}$ as a fraction.

Solution We cannot express this number as the infinite sum of a geometric sequence. This number should be written so that the nonrepeating part is not included inside the sequence.

$$\begin{aligned} 2.1\overline{5} &= 2.1555... \\ &= \underbrace{2.1}_{\text{nonrepeating part}} + \underbrace{0.05 + 0.005 + 0.0005 + ...}_{\text{infinite sum of a geometric sequence}} \\ &= 2.1 + 0.05 + 0.05 \cdot 0.1 + 0.05 \cdot (0.1)^2 + ... \qquad (b_1 = 0.05,\ q = 0.1) \end{aligned}$$

Therefore, $2.1\overline{5} = 2.1 + \dfrac{0.05}{1-0.1} = \dfrac{21}{10} + \dfrac{5}{90} = \dfrac{97}{45} = 2\dfrac{7}{45}.$

3. Equations with Infinitely Many Terms

EXAMPLE 87 Solve $2 + 2x + 2x^2 + ... = \dfrac{4}{x}$.

Solution In this problem our traditional methods of solving equations will not help since we cannot see completely which equation we have. Let us try to see an infinite sum of a geometric sequence in this equation.

$$\underbrace{2}_{b_1} + \underbrace{2\cdot \overset{q}{x}}_{b_2} + \underbrace{2\cdot x^2}_{b_3} + \ldots = \underbrace{\frac{4}{x}}_{S}$$

Here, we should note that this equation will have a solution if and only if $|q| < 1$, that is $|x| < 1$. If $|x| > 1$, there is no infinite sum.

Now, using the infinite sum formula,

$S = \dfrac{b_1}{1-q}$, that is $\dfrac{4}{x} = \dfrac{2}{1-x}$, so $x = \dfrac{2}{3}$.

Since $\left|\dfrac{2}{3}\right| < 1$, the only solution of this non-standard equation is $x = \dfrac{2}{3}$.

EXAMPLE 88 Solve $2x+1+x^2-x^3+x^4-x^5+\ldots=\dfrac{13}{6}$.

Solution Now we have:

$$2x+1+\overbrace{\underbrace{x^2}_{b_1} + \underbrace{(-x^3)}_{b_2} + \underbrace{x^4}_{b_3} + \underbrace{(-x^5)}_{b_4} + \ldots}^{S} = \frac{13}{6} \qquad (q=-x,\ \ |x|<1)$$

Note that since there is no way to express $2x + 1$ in the infinite sum, we exclude it from the geometric sequence.

Now, using the infinite sum formula,

$2x+1+\dfrac{x^2}{1-(-x)} = \dfrac{13}{6}$, which means $18x^2+5x-7=0$.

Solving the quadratic equation gives $x = -\dfrac{7}{9}$ or $x = \dfrac{1}{2}$, both of which satisfy the condition $|x| < 1$. So our answer is $x = -\dfrac{7}{9}$ or $x = \dfrac{1}{2}$.

Check Yourself 13

1. Can we find $\dfrac{1}{2}+\dfrac{3}{4}+\dfrac{9}{8}+\ldots$? Why?
2. Find $\dfrac{9}{10}-\dfrac{9}{10^2}+\dfrac{9}{10^3}-\ldots$.
3. Write $0.0\overline{6}$ as a fraction.
4. Solve $x+x^3+x^5+\ldots=\dfrac{3}{8}$.

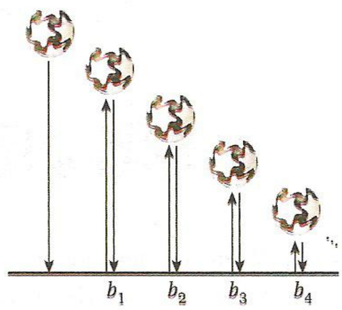
Answers

1. no, because $q > 1$ 2. $\frac{9}{11}$ 3. $\frac{1}{15}$ 4. $\frac{1}{3}$

4. Applied Problems

EXAMPLE 89 A ball is dropped from a height of 50 cm. Each time it bounces, it returns to $\frac{1}{3}$ of its previous height. How far will the ball travel in the air before coming to rest?

Solution This example is very similar to Example 81. The only difference is that we are not looking for a finite sum, such as S_5. Since we are sure that the ball will stop ($q < 1$), the required distance, say S, can be expressed as follows:

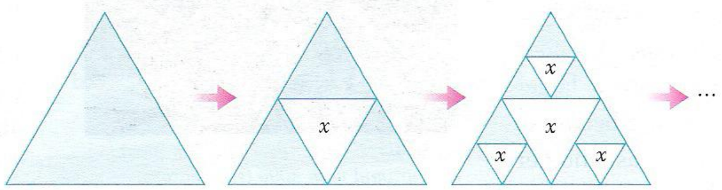


$$S = \underbrace{50}_{\substack{\uparrow \\ \text{first fall, not part} \\ \text{of a geometric} \\ \text{sequence}}} + \underbrace{2}_{\substack{\uparrow \\ \text{(the ball covers each} \\ \text{distance twice, as in} \\ \text{the diagram)}}} \cdot \frac{50 \cdot \frac{1}{3}}{1 - \frac{1}{3}} = 100 \text{ cm.}$$

So the ball will travel 100 cm before coming to rest.

EXAMPLE 90 Consider an equilateral triangle made from paper. We take our scissors and cut off smaller equilateral triangles from the original triangle using the following principle: *connect the middle points of the sides of every triangle you see. Cut out and throw away the middle triangle you make. Repeat the process with every new triangle you see.* How much of the original area will remain if we don't stop cutting?

Solution Let us look at a diagram of the problem, where x shows the area of the triangle we throw away each time:



After cutting the first triangle, we throw away one new triangle. After cutting the second triangle we throw away three new triangles, and so on.

Now let a be the sidelength of our equilateral triangle. If we say S is the area of the triangle at the beginning, and S' is the sum of the subtracted areas we have,

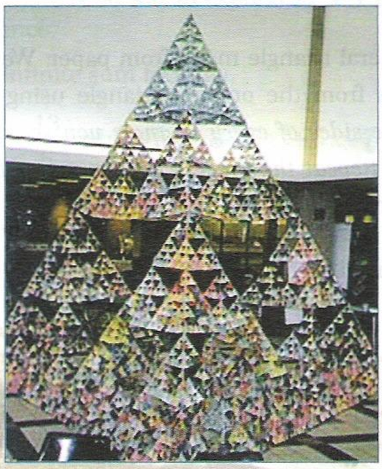
$S = \frac{a^2\sqrt{3}}{4}$ (formula for area of an equilateral triangle with sidelength a)

$$S' = \underbrace{1}_{\substack{\text{we cut out}\\ \text{one triangle}\\ \text{in the first phase}}} \cdot \frac{\overbrace{\left(\frac{a}{2}\right)}^{\substack{\text{sidelength of}\\ \text{the triangle}\\ \text{that we cut}}}{}^{2} \cdot \sqrt{3}}{4} + 3 \cdot \frac{\left(\frac{a}{2^2}\right)^2 \cdot \sqrt{3}}{4} + 9 \cdot \frac{\left(\frac{a}{2^3}\right)^2 \cdot \sqrt{3}}{4} + \ldots = \frac{\frac{a^2\sqrt{3}}{16}}{1-\frac{3}{4}} = \frac{a^2\sqrt{3}}{4}.$$

Clearly, $S - S' = 0$. Therefore, no area will remain if we don't stop cutting.

THE SIERPINSKI PYRAMID

The problem we have just looked at is similar to the construction of a special structure in mathematics called a **Sierpinski Pyramid**. A Sierpinski Pyramid begins with a single tetrahedron, i.e. a pyramid whose sides are all identical equilateral triangles. Each tetrahedron is placed on its triangular base, then raised and set so that its bottom three vertices meet the top of three other tetrahedrons. Together these four tetrahedrons create the form of a larger tetrahedron. This can be treated as a single unit, and become part of an even greater tetrahedron. This process of combining four tetrahedrons to create a larger one demonstrates the beauty of math, but it also has another purpose: it shows exponential growth.



In the picture there is a Sierpinski Pyramid with 4096 tetrahedrons.

In this case the growth would be in terms of 4 to the power x. The original tetrahedron is described as level zero because it is 4 to the power 0 which is equal to 1. The next grouping is described as level one because it is 4 to the power 1, which is equal to four tetrahedrons.

As the process of building more and more levels continues, the number of tetrahedrons on each level increases by a power of four:

Level 2: 4 to the power 2 (4^2)

Level 3: 4 to the power 3 (4^3)

Level 4: 4 to the power 4 (4^4)

Level 5: 4 to the power 5 (4^5)

Level 6: 4 to the power 6 (4^6) tetrahedrons, i.e.

4096 individual tetrahedrons.

EXERCISES 3

A. Geometric Sequences

1. State whether the following sequences are geometric or not.

   a. $(2, -5, \frac{25}{2}, ...)$ b. $(b_n) = (4^{n^2-3})$

   c. $(b_n) = (2n + 7)$

2. Find the general term of the geometric sequence with the given qualities.

   a. $b_1 = 5, q = 2$ b. $b_1 = -3, q = \frac{1}{2}$

   c. $b_1 = 1000, q = \frac{1}{10}$ d. $b_1 = \sqrt{3}, q = \sqrt{3}$

   e. $b_1 = 4, b_4 = 32$ f. $b_1 = 3, b_5 = \frac{1}{27}$

   g. $b_3 = 32, b_6 = \frac{1}{2}$ h. $b_5 = 5, b_{25} = 5$

   i. $b_1 = 2, b_6 = 8\sqrt{2}$

3. Fill in the blanks to form a geometric sequence.

   a. $3 - 2\sqrt{2}$, __ , $3 + 2\sqrt{2}$

   b. __ , __ , 36, __ , 4

4. Find the general term of the geometric sequence with $b_4 = b_2 + 24$ and $b_2 + b_3 = 6$.

5. Write the first four terms of the non-monotone geometric sequence that is formed by inserting nine terms between –3 and –729.

6. Given a geometric sequence with $b_6 = 4b_4$ and $b_3 \cdot b_6 = 1152$, find b_1.

7. The thirteenth and seventeenth terms of a geometric sequence are $\frac{1}{4}$ and 48 respectively. Find the product of the fourteenth and sixteenth terms.

8. The sixth and eighth terms of a geometric sequence are $\frac{\sqrt{n}+3}{\sqrt{n}+1}$ and $\frac{n-1}{25n-75+50\sqrt{n}}$ respectively. Find the seventh term.

9. The sum of the first two terms of a monotone geometric sequence is 15. The first term exceeds the common ratio by $\frac{25}{3}$. Find the fourth term of this sequence.

10. Given a non-monotone geometric sequence with $\frac{b_4}{b_6} = \frac{1}{4}$ and $b_2 + b_5 = 216$, find b_1.

11. Can the numbers 10, 11, 12 be terms (not necessarily consecutive) of a geometric sequence?

B. Sum of the Terms of a Geometric Sequence

12. For each geometric sequence (b_n) find the missing value.

    a. $b_1 = -\frac{3}{2}, q = -2, S_7 = ?$

    b. $b_2 = 6, b_7 = 192, S_{11} = ?$

    c. $b_2 = 1, b_5 \cdot b_2 = 64 \cdot b_4 \cdot b_5, S_5 = ?$

    d. $S_3 = 111, q^3 = 4, S_6 = ?$

13. The general term of a geometric sequence is $b_n = 3 \cdot 2^n$. Find S_{10}.

14. The general term of a geometric sequence is $b_n = \left(\frac{5}{3}\right)^{n-1}$. Find the formula for S_n.

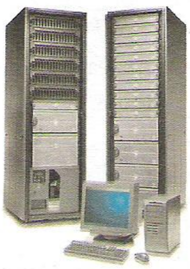
15. Find the common ratio of a geometric sequence if $\frac{S_4}{S_2} = \frac{5}{4}$.

16. The sum of the first four terms of a geometric sequence is 20 and the sum of the next four terms is 320. Find the sum of the first twelve terms.

17. A chain letter is sent to five people. Each of the five people mails the letter to five other people, and the process is repeated. What is the total number of people who have received the letter after four mailings?

18. You want to paint the wood around four windows in your house. You think that you can paint each window in 90% of the time it took to paint the previous window. If it takes you thirty minutes to paint the first window, how long will it take to paint all four windows?

19. A computer solved several problems in succession. The time it took the computer to solve each successive problem formed a geometric sequence. How many problems did the computer solve if it took 63.5 minutes to solve all the problems except the first, 127 minutes to solve all the problems except the last, and 31.5 minutes to solve all the problems except for the first two?



20. Show that $(\underbrace{66\ldots6}_{n\ digits})^2 + \underbrace{88\ldots8}_{n\ digits} = \underbrace{44\ldots4}_{2n\ digits}$.

C. Infinite Sum of a Geometric Sequence *(Optional)*

21. For each geometric sequence (b_n) find the missing value.

a. $q = \frac{1}{2}, S = 12, b_1 = ?$ b. $b_1 = \frac{5}{2}, S = \frac{3}{2}, q = ?$

c. $S = 5b_1, q = ?$

22. Find the infinite sums.

a. $54 + 18 + 6 + \ldots$ b. $\frac{3}{2} - 1 + \frac{2}{3} - \ldots$

c. $7 + 3 + \frac{9}{7} + \ldots$ d. $\frac{1}{12} - \frac{1}{6} + \frac{1}{3} - \ldots$

23. Write each repeating decimal as fraction.

a. $0.\overline{21}$ b. $5.1\overline{42}$ c. $-3.\overline{202}$ d. $2.0\overline{65}$

24. Find $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \ldots$

25. (b_n) is a geometric sequence with infinite sum 243 and $S_5 = 275$. Write the first four terms of this sequence.

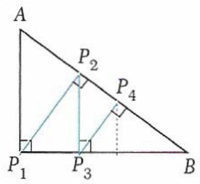
26. A square has sides of length 1 m. A man marks the midpoints on each side of the square and joins them to create a second square, inside the first square. He then repeats the process to create a third square inside the second, and so on. If the man never stops, find:

a. the sum of the perimeters of all the squares.

b. the sum of the areas of all the squares.

27. The bob of a pendulum swings through an arc 30 cm long on its first swing. Each successive swing is $\frac{4}{5}$ of the length of the preceding one. Find the total distance that the bob travels before it stops.

28. Let AP_1B be a right triangle where $\angle AP_1B = 90°$. The line P_1P_2 is drawn from P_1, and another is drawn in triangle BP_1P_2, and so on. Find the sum of the length of all drawn lines $(P_1P_2 + P_2P_3 + P_3P_4 + \ldots)$ if $AP_1 = 3$ and $BP_1 = 4$.



29. (b_n) is a geometric sequence with infinite sum, 3 and $b_1^3 + b_2^3 + \ldots = \frac{108}{13}$. Find b_1 and q.

30. Solve $1 + \left(\frac{x+1}{x-1}\right) + \left(\frac{x+1}{x-1}\right)^2 + \ldots = \frac{x^2}{2}$.

31. Solve $x^{-2} + x^{-4} + \ldots = 0.125$ if $1 + \frac{x}{4} + \frac{x^2}{16} + \ldots < 1$.

32. Given $|x| < 1$, simplify $1 + 2x + 3x^2 + 4x^3 + \ldots$

Mixed Problems

33. $5x - y$, $2x + 3y$, $x + 2y$ form an arithmetic sequence. $(y + 1)^2$, $xy + 1$, $(x - 1)^2$ form a geometric sequence. Find x and y.

34. The first, the third, and the fifth term of a geometric sequence are equal to the first, the fourth, and the sixteenth term of a certain arithmetic sequence respectively. Find the fourth term of the arithmetic sequence if its first term is 5.

35. Three numbers form an arithmetic sequence. If we add 8 to the first number, we get a geometric sequence with the sum of terms equal to 26. Find the three numbers.

36. (a_n) is an arithmetic sequence with non-zero common difference. $a_1 \cdot a_2$, $a_2 \cdot a_3$, $a_3 \cdot a_1$ form a geometric sequence. Find the common ratio of the sequence.

37. x, y, z form an arithmetic sequence and y, z, t form a geometric sequence such that $x + t = 21$, $z + y = 18$. Find x, y, z, t.

38. Prove that the product of the first n terms of a geometric sequence (b_n) is $(b_1 \cdot b_n)^{\frac{n}{2}}$.

39. A teacher wrote the numbers –2,7 on the blackboard and told the students that these are the first two terms of a sequence. He asked the students to find the third term. Since he didn't mention the type of the sequence, some students thought the sequence was arithmetic while some thought it was geometric. Find the positive difference between the two possible answers to the teacher's problem.

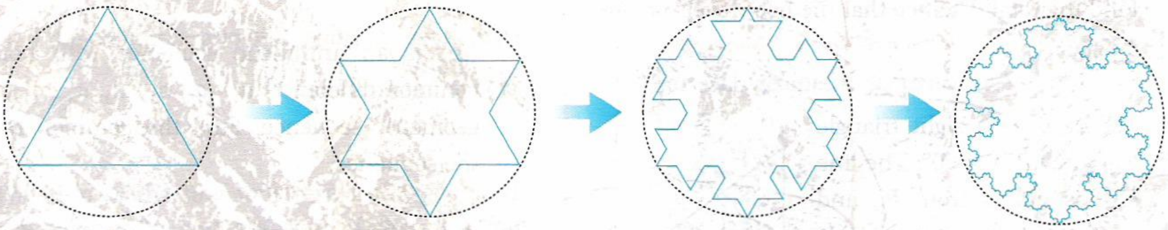
40. The arithmetic mean of 2 and a number is two less than twice their geometric mean. Find the number.

41. The numbers x, y, z form a geometric sequence such that $x + y + z = 26$. If $x + 1$, $y + 6$, $z + 3$ form an arithmetic sequence, find x, y, z.

42. If $\frac{1}{b-a}$, $\frac{1}{2b}$, $\frac{1}{b-c}$ form an arithmetic sequence, show that a, b, c form a geometric sequence.

43. Find $\sqrt{\underbrace{11\ldots11}_{16\ digits} - \underbrace{22\ldots22}_{8\ digits}}$.

THE KOCH SNOWFLAKE



A **Koch snowflake** is another mathematical construction. We make a Koch snowflake by making progressive additions to an equilateral triangle. We divide the triangle's sides into thirds, and then create a new triangle on each middle third. Then we repeat the process over and over. Thus, each snowflake shows more complexity, but every new triangle in the design looks exactly like the initial one.

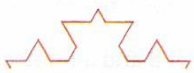
Now imagine drawing a circle around the original figure. Notice that no matter how large the perimeter gets, the area of the figure remains inside the circle. **In the Koch Snowflake, an infinite perimeter encloses a finite area.** Although it sounds impossible, we can prove it as follows:

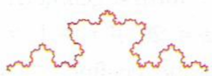
Calculating the perimeter of the Koch Snowflake:

To simplify the problem, let us describe what happens to one side of the triangle as the procedure is repeated. Suppose that the original length of one side is L. Then we go through the following steps:

Step 1:  one segment of length L.

Step 2:  four segments, each of length $\frac{L}{3}$. The total length of the side is now $\frac{4}{3}L$.

Step 3:  four times four segments, each of length $\left[\frac{1}{3} \cdot \frac{L}{3}\right]$. The total length of the side is now $\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)L = \left(\frac{4}{3}\right)^2 L.$

Step n:  Total length $= \left(\frac{4}{3}\right)^{n-1} L.$

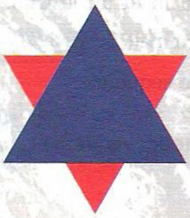
At each stage of the process, the length of one of the original sides of the triangle increases by a factor of $\frac{4}{3}$. Considering that we measure this length three times for each snowflake (as each snowflake has three sides), this leads to a geometric sequence of the form $3L\left(\frac{4}{3}\right)^{n-1}$. Since $q > 1$, the sequence grows without bound. Thus, the perimeter of the Koch snowflake is infinite.

Calculating the area of the Koch Snowflake:

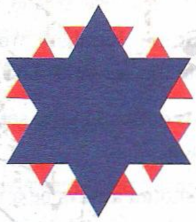
Suppose that the area of the original triangle is A.

Step 1: Total area is A.

Step 2: Total area is $A+3\left(\frac{A}{9}\right)=A\left(1+\frac{3}{9}\right)$

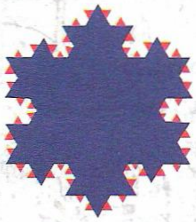


Step 3: Total area is $A(1+\frac{3}{9}+3\cdot 4\cdot\frac{1}{9}\cdot\frac{1}{9})$



Step 4: Total area is

$$A(1+\frac{3}{9}+3\cdot 4\cdot\frac{1}{9}\cdot\frac{1}{9}+3\cdot 16\cdot\frac{1}{9}\cdot\frac{1}{9}\cdot\frac{1}{9})$$



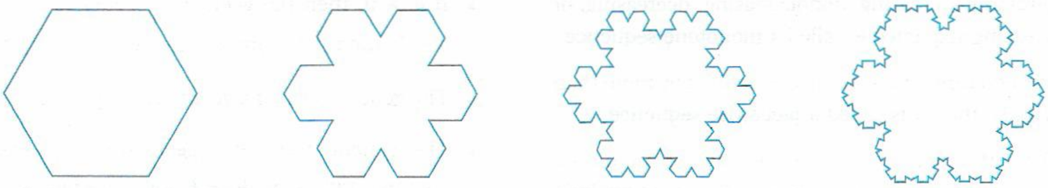
Step n: Total area is $A(1+\frac{3}{9}+3\cdot 4\cdot(\frac{1}{9})^2+3\cdot 4^2\cdot(\frac{1}{9})^3+...+3\cdot 4^{n-2}\cdot(\frac{1}{9})^{n-1})$

Note that after $\frac{3}{9}$ each term in this sum is $\frac{4}{9}$ times the previous one. Therefore we can calculate the sum of all the areas added using the formula for the sum of an infinite geometric sequence: Area $=A\left(1+\frac{\frac{3}{9}}{1-\frac{4}{9}}\right)=\frac{8A}{5}$.

This is the area of the entire snowflake, which means that even if we repeat this procedure without end, the total area will never be more than $\frac{8A}{5}$.

If we combine our calculations of the perimeter and area of the snowflake, we have proved that an infinite perimeter borders a finite area.

Below is another kind of snowflake. What can you say about its area and perimeter?



Try producing your own snowflakes.

CHAPTER SUMMARY

1. Real Number Sequences

- By the set of natural numbers we mean all positive integers and denote this set by $\mathbb{N}$. That is, $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- A function which is defined in the set of natural numbers is called a **sequence.**
- In a sequence, n should always be a natural number, but the value of a_n may be any real number depending on the formula for general term.
- For an arithmetic sequence (a_n),

  a_1 is the first term,

  a_2 is the second term,

  a_3 is the third term,

  $\vdots$

  a_n is the nth term or **general term**.

  $(a_n) = (a_1, a_2, a_3, \ldots, a_n, \ldots)$
- (a_n) represents a sequence, a_n represents its general term.
- If there is at least one natural number which makes the general term of a sequence undefined, then the given function is not a sequence.
- If a sequence contains a countable number of terms, then we say it is **finite**, otherwise it is **infinite**.
- If each term of a sequence is greater than the previous term, then the sequence is called an **increasing sequence.**
- If each term of a sequence is less than the previous term, then the sequence is called a **decreasing sequence.**
- In general any increasing, nondecreasing, decreasing, or nonincreasing sequence is called a **monotone sequence**.
- If the general term of a sequence is defined by more than one formula, then it is called a **piecewise sequence.**
- Sometimes the terms in a sequence may depend on the other terms. Such a sequence is called a **recursively defined sequence.**

2. Arithmetic Sequences

- If a sequence (a_n) has the same difference d between its consecutive terms, then it is called an **arithmetic sequence.** That is, (a_n) is arithmetic if $a_{n+1} = a_n + d$ and $d \in \mathbb{R}$. d is called the **common difference.**
- If $d > 0$, then the arithmetic sequence is **increasing**. If $d < 0$, then the arithmetic sequence is **decreasing**.
- The general term of an arithmetic sequence is linear.
- The general term of an arithmetic sequence (a_n) with common difference d is $a_n = a_p + (n - p)d$, where a_p is any term of the sequence.
- In an arithmetic sequence $a_p = \dfrac{a_{p-k} + a_{p+k}}{2}$, where $k < p$.
- Three numbers a, b, c form consecutive terms of an arithmetic sequence if and only if $b = \dfrac{a+c}{2}$. b is called the **arithmetic mean** of a and c.
- The sum of the first n terms of an arithmetic sequence (a_n) is $S_n = \dfrac{a_1 + a_n}{2} \cdot n$.

3. Geometric Sequences

- If a sequence (b_n) has the same ratio q between its consecutive terms, then it is called a **geometric sequence.** That is, b_n is geometric if $b_{n+1} = b_n \cdot q$, where $q \in \mathbb{R}$. q is called the **common ratio.**
- If $q > 0$, then the geometric sequence is **monotone**. If $q < 0$, then the geometric sequence is **not monotone**.
- The general term of a geometric sequence is exponential.
- The general term of a geometric sequence (b_n) with common ratio q is $b_n = b_p \cdot q^{n-p}$, where b_p is any term of the sequence.

- In a geometric sequence $b_p^2 = b_{p-k} \cdot b_{p+k}$, where $k < p$.
- Three numbers a, b, c form consecutive terms of a geometric sequence if and only if $b^2 = a \cdot c$. We call b the **geometric mean** of a and c.
- The sum of the first n terms of a geometric sequence (b_n) with common ratio q is $S_n = b_1 \cdot \dfrac{1-q^n}{1-q}$ $(q \neq 1)$.
- The infinite sum of a geometric sequence (b_n) with common ratio q is $S = \dfrac{b_1}{1-q}$ $(|q| < 1)$.
- If $|q| \geq 1$, then the geometric sequence has no infinite sum.
- Numbers such as 0.66666... are called **repeating decimals** and denoted as $0.\overline{6}$ or 0.(6).

Concept Check

- What is the difference between a_n and (a_n)?
- When can a given formula not be the general term of a sequence?
- Is it possible to find the $\sqrt{2}$th term of a sequence?
- Is it possible that a term of a sequence is $\sqrt{2}$?
- Why are recursively defined sequences not very practical?
- What is an arithmetic sequence?
- When is an arithmetic sequence increasing?
- What is the nth term of an arithmetic sequence (a_n) if $a_x = x$ and $a_y = y$?
- How many variables do we need to know to find common difference in an arithmetic sequence? What are these variables?
- Which condition must be satisfied for three numbers to form an arithmetic sequence?
- If S_n is the sum of the first n terms of an arithmetic sequence, is it possible that S_1, S_2, S_3, ... is also an arithmetic sequence? If so, give an example.
- What is a geometric sequence?
- When is a geometric sequence decreasing?
- Can the common ratio of a geometric sequence be 0? Why?
- Which condition must be satisfied for three numbers to form a geometric sequence?
- How can we find the sum of the first n terms of a geometric sequence with common ratio 1?
- When can we find the infinite sum of a geometric sequence?
- A turtle is 10 m away from a tree. Every hour, the turtle walks half the distance to the tree. How long will it take for the turtle to walk to the tree?
- What kind of problems can be solved with the help of the infinite sum of geometric sequences?

CHAPTER REVIEW TEST 1

1. Which terms can be the general term of a sequence?

I. $\frac{n}{n-2}$ II. 3 III. $n^2 + 2n + 3$

IV. $\sqrt{7-n}$ V. 3^n VI. n^n

A) I, II, III, IV B) II, III, IV, VI
C) I, II, III, VI D) II, III, V, VI
E) III, IV, V, VI

2. Which of the following can be the general term of the sequence with the first four terms 3, 5, 7, 9?

A) $2n - 1$ B) $2n$ C) $2n + 1$
D) $n + 1$ E) $n^2 + 2$

3. Given $a_1 = 2$, and $a_{n+1} = \frac{2a_n + 5}{2}$ for $n \geq 1$, find a_{11}.

A) 27 B) 25 C) 22
D) $\frac{27}{2}$ E) $\frac{25}{2}$

4. How many terms of the sequence with general term $\frac{n^2 - 2n + 36}{n}$ are natural numbers?

A) 5 B) 6 C) 7 D) 8 E) 9

5. How many terms of the sequence with general term $a_n = (\frac{2n+1}{n+9})$ are less than $\frac{1}{3}$?

A) 0 B) 1 C) 2 D) 3 E) 4

6. Given $a_n = (\frac{3n^2 - 5n}{n+k-3})$ and $a_5 = 3$, find k.

A) 3 B) 5 C) $\frac{22}{3}$
D) $\frac{35}{3}$ E) $\frac{44}{3}$

7. How many of the following sequences are decreasing?

I. $(a_n) = (\frac{3n-5}{n+2})$ II. $(b_n) = (n-3)^2$

III. $(c_n) = (-1^n)$ IV. $(d_n) = (\frac{1}{n+1})$

V. $(e_n) = (\frac{(-1)^n}{3n+2})$

A) 1 B) 2 C) 3 D) 4 E) 5

8. What is the minimum value in the sequence formed by $a_n = (\frac{2n+3}{3n-7})$?

A) –1 B) –3 C) –2 D) –7 E) –8

9. Which one of the following is the general term of an arithmetic sequence?

A) $n^2 + 2n$ B) $4n + 5$ C) n^3
D) $2^n + 3$ E) 5^n

10. If $\frac{1}{3}$, a, b, c, $\frac{5}{8}$ are consecutive terms of an arithmetic sequence, find $a + b + c$.

A) $\frac{7}{24}$ B) $\frac{23}{24}$ C) $\frac{21}{16}$ D) $\frac{23}{16}$ E) $\frac{69}{49}$

11. (a_n) is an arithmetic sequence with $a_{11} = 8$ and $a_{20} = 35$. Find a_3.

A) –3 B) –6 C) –16 D) –22 E) –28

12. (a_n) is arithmetic sequence with $a_1 = 7$ and common difference $\frac{1}{3}$. Find the general term.

A) $3n + 4$ B) $\frac{n+7}{3}$ C) $\frac{n-4}{3}$
D) $\frac{n+4}{3}$ E) $\frac{n+20}{3}$

13. (a_n) is an arithmetic sequence such that $a_3 + a_4 = 23$ and $a_5 + a_4 = 37$. Find a_8.

A) 49 B) 47 C) 45 D) 44 E) 43

14. (a_n) is a finite arithmetic sequence with first term $\frac{1}{2}$, last term $\frac{1}{16}$, and sum 9. How many terms are there in this sequence?

A) 9 B) 16 C) 32 D) 48 E) 64

15. $x - 2$, $x + 8$, $3x + 2$ form an arithmetic sequence. Find x.

A) 12 B) 11 C) 10 D) 9 E) 8

16. (a_n) is an arithmetic sequence with $S_4 = 3(S_4 - S_7)$ and $a_1 = 1$. Find the common difference.

A) $-\frac{2}{51}$ B) $-\frac{13}{51}$ C) $\frac{2}{51}$
D) $\frac{13}{51}$ E) $\frac{15}{51}$

CHAPTER REVIEW TEST 2

1. The sum of the first three terms of an arithmetic sequence is 33 and the sum of the first 33 terms is 3333. Find the sum of the first ten terms.

A) 320 B) 330 C) 360 D) 630 E) 660

2. (a_n) is an arithmetic sequence such that $S_{13} = 195$ and $a_{13} - a_1 = 24$. Find a_1.

A) 2 B) 3 C) 4 D) 5 E) 6

3. How many of the following sequences are geometric?

I. $(b_n) = (2^n)$ II. $(b_n) = (4^{3n})$

III. $(b_n) = ((n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1)$

IV. $(b_n) = (2n+1)$ V. $(b_n) = (5 \cdot 3^{n-1})$

A) 0 B) 1 C) 2 D) 3 E) 4

4. $\frac{3}{7}$, a, b, c, $-\frac{1}{35}$ form an arithmetic sequence. Find $\frac{a-b}{c}$.

A) $\frac{2}{3}$ B) $\frac{4}{3}$ C) $-\frac{2}{3}$ D) $\frac{3}{4}$ E) $\frac{1}{2}$

5. (b_n) is a geometric sequence with fourth term $\frac{1}{8}$ and tenth term $\frac{1}{32}$. Find the seventh term.

A) $\frac{1}{32}$ B) $\frac{1}{16}$ C) $\frac{1}{8}$ D) $\frac{1}{4}$ E) $\frac{1}{2}$

6. (b_n) is a geometric sequence with first term $\frac{1}{7}$ and common ratio $\frac{1}{2}$. Find the general term.

A) $\frac{2}{7} \cdot 2^n$ B) $\frac{1}{14} \cdot 2^n$ C) $\frac{2}{7 \cdot 2^n}$ D) $\frac{7}{2 \cdot 2^n}$ E) $\frac{14}{2^n}$

7. (b_n) is a geometric sequence such that $b_3 - b_4 = \frac{16}{9}$ and $b_6 - b_8 = \frac{1}{3}$. Which one of the following can be the common ratio?

A) $\frac{1}{2}$ B) $\frac{2}{3}$ C) 1 D) $\frac{3}{2}$ E) 2

8. Seven numbers are inserted between 16 and $\frac{1}{16}$ to form a monotone geometric sequence. Find the fourth term of this sequence.

A) 4 B) 2 C) $\frac{1}{2}$ D) $\frac{1}{4}$ E) $\frac{1}{8}$

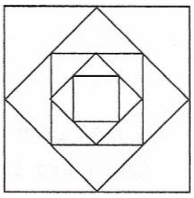
9. $\dfrac{625}{32}, \dfrac{125}{16}, \dfrac{25}{8}$ are the first three terms of a geometric sequence. Find the eighth term.

A) $\dfrac{125}{8}$ B) $\dfrac{25}{16}$ C) $\dfrac{16}{25}$ D) $\dfrac{8}{625}$ E) $\dfrac{4}{125}$

10. A ball is dropped from a height of 243 m. Every time it hits the ground, it bounces back to $\dfrac{1}{3}$ of its previous height. What is the height of the ball at the peak of its tenth bounce?

A) $\dfrac{1}{9}$ m B) $\dfrac{1}{27}$ m C) $\dfrac{1}{81}$ m D) $\dfrac{1}{243}$ m E) $\dfrac{1}{486}$ m

11. In the figure the largest square has sides of length six units. Each subsequent square connects the midpoints of the sides of the previous square. What is the perimeter of the ninth square in the diagram?



A) $\dfrac{3}{2}$ B) $\dfrac{3\sqrt{2}}{2}$ C) 3 D) $3\sqrt{2}$ E) $6\sqrt{2}$

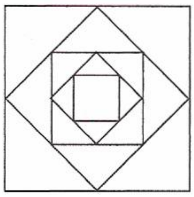
12. The numbers $x - 3$, 3, $y + 5$ form both an arithmetic and a geometric sequence. Find $x - y$.

A) 0 B) 2 C) 4 D) 8 E) 16

13. A ball is dropped from a height of 10 m. Every time it hits the ground, it bounces back to half of its previous height. What is the total distance that the ball has traveled when it stops?

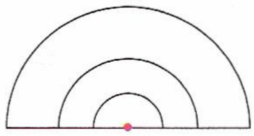
A) 15 m B) 20 m C) 30 m D) 40 m E) 60 m

14. In the figure the largest square has sides of length six units. Each subsequent square connects the midpoints of the sides of the previous square. The process continues infinitely. Find the difference between the total perimeter of all the squares and the total area of all the squares, as a numerical value.



A) $24(2 + \sqrt{2})$ B) $24(2 - \sqrt{2})$ C) $24(\sqrt{2} - 1)$ D) $24(\sqrt{2} + 1)$ E) $24(2 - \sqrt{2})$

15. In the figure the largest semicircle has radius 4 cm. A semicircle is drawn inside this semicircle with the same center but half the radius. If this process is repeated without end, what is the total area of all the semicircles?



A) $\dfrac{16\pi}{3}$ cm^2 B) $\dfrac{32\pi}{3}$ cm^2 C) 32π cm^2 D) $\dfrac{64\pi}{3}$ cm^2 E) 64π cm^2

16. Find $\dfrac{1}{3}-\dfrac{1}{2}+\dfrac{1}{9}-\dfrac{1}{4}+\dfrac{1}{27}-\dfrac{1}{8}+\dots$

A) -1 B) $-\dfrac{1}{2}$ C) 0 D) $\dfrac{1}{2}$ E) 1

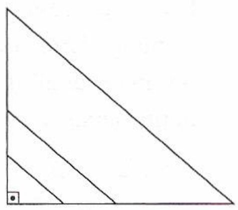
CHAPTER REVIEW TEST 3

1. Given $y > x > 0$, simplify $x^2 + \frac{x^3}{y^4} + \frac{x^4}{y^5} + \frac{x^5}{y^6} + \ldots$

A) $\frac{x^3}{y^4 - y^3x}$ B) $x^2 - \frac{x^3}{y^4 - y^3x}$ C) 0

D) $\frac{x^2 + x^3}{y^4 - y^3x}$ E) $x^2 + \frac{x^3}{y^4 - y^3x}$

2. In the figure the right sides of the largest triangle have lengths three units and four units respectively. Each subsequent triangle joins the midpoints of the sides of the previous triangle. This process continues infinitely. What is the total area of all the triangles?



A) 16 B) 12 C) 10 D) 8 E) 4

3. Which one of the following is the fraction form of $0.1\overline{3}$?

A) $\frac{2}{15}$ B) $\frac{13}{90}$ C) $\frac{1}{75}$ D) $\frac{11}{90}$ E) $\frac{13}{99}$

4. How many terms of the sequence with general term $a_n = \frac{2n - 13}{3n + 7}$ are negative?

A) 9 B) 8 C) 7 D) 6 E) 5

5. (a_n) is a sequence such that $a_{n+1} + a_n \cdot n - 3 = (n - 3) \cdot a_n$, and $a_2 = 7$. Find a_4.

A) 39 B) 57 C) 75 D) 93 E) 107

6. $5 - \sqrt{5}$, x, $5 + \sqrt{5}$ form a monotone geometric sequence. Find the common ratio.

A) $2 + \sqrt{5}$ B) $\frac{\sqrt{5}+1}{2}$ C) $\frac{10\sqrt{5} - 10}{5 + \sqrt{5}}$

D) $\frac{10 - 10\sqrt{5}}{5 - \sqrt{5}}$ E) $2 - \sqrt{5}$

7. (b_n) is a geometric sequence with first term 4 and eighth term 25. Find the product of the first eight terms.

A) 10^6 B) 10^7 C) 10^8 D) 10^{10} E) 10^{12}

8. Twelve numbers are inserted between 16 and 81 to form an arithmetic sequence. What is the sum of the twelve numbers?

A) 682 B) 679 C) 582 D) 579 E) 485

9. Given an arithmetic sequence with $S_n = n(2n + 7)$, find the general term.

A) $4n + 3$ B) $4n + 5$ C) $5n - 4$ D) $4n - 13$ E) $5n + 13$

10. (c_n) is an arithmetic sequence with $c_a = b$ and $c_b = a$. The sum of the first seven terms is 7. Find c_3.

A) -3 B) -1 C) 0 D) 2 E) 4

11. (b_n) is a geometric sequence with third term a and sixth term $16a^5$. Find the first term.

A) $2^{\frac{4}{3}} \cdot a^{\frac{3}{2}}$ B) $2^{-\frac{8}{3}} \cdot a^{-\frac{5}{3}}$ C) $2^{-\frac{4}{3}} \cdot a^{\frac{5}{2}}$ D) $2^{\frac{3}{8}} \cdot a^{-\frac{5}{2}}$ E) $2^{-\frac{8}{3}} \cdot a^{-\frac{5}{2}}$

12. a terms are inserted between $1 + a$ and $a^3 + 1$ to form an arithmetic sequence. Find the common difference of the sequence.

A) $a^2 + a$ B) $a^2 - a$ C) $-a^2 - 1$ D) $a^2 - 1$ E) $a - 1$

13. (a_n) is an increasing arithmetic sequence with positive terms. The sum of a_6, a_7 and a_8 is 36 and the sum of the squares of these terms is 450. Find the nineteenth term.

A) 39 B) 42 C) 48 D) 49 E) 54

14. The roots of the equation $3x^3 + 9x^2 + 2x - a = 0$ form an arithmetic sequence. Find a.

A) -4 B) -2 C) -1 D) 2 E) 4

15. Solve $1 + x + x^2 + ... = x + 3$.

A) $\sqrt{3}$ B) $-\sqrt{3} - 1$ C) $\sqrt{3} - 1$ D) $2 - \sqrt{3}$ E) $\sqrt{3} - 2$

16. The interior angles of a quadrilateral form a geometric sequence such that the first term is four times the third term. Find the greatest angle.

A) 196° B) 192° C) 186° D) 182° E) 176°

ANSWERS TO EXERCISES

EXERCISES 1

1. a. yes b. yes c. yes d. no e. yes f. yes g. no h. yes i. no 2. a. $2n-1$ b. $(-1)^n(2n-1)$ c. n^2-1 d. $-\frac{n^3}{2n+3}$ e. n^2+n

3. a. 5, 7, 9; 77 b. $\frac{1}{2}, \frac{7}{9}, 1; \frac{5}{2}$ c. $\sqrt{7}, 4, 3\sqrt{3}; 6\sqrt{2}$ 4. 7 5. 3 6. 3 7. $(n-1)(n-2)(n-3)(n-4)+2n$; 132

8. 58 9. $\begin{cases} n+1 & ,n \text{ odd} \\ n-1 & ,n \text{ even} \end{cases}$ 10. find the difference of consecutive terms 11. monotone 12. a. a_2 smallest b. b_2 biggest c. c_1 smallest 13. a. 1, 2, 4, 8; 2^{n-1} b. –3, 2, 7, 12; $5n-8$ c. 3, 15, 105, 945; not possible

14. a. $a_1 = 3$, $a_n = a_{n-1} + 3$ b. $b_1 = 2$, $b_n = 2 \cdot b_{n-1}$ c. $c_1 = -4$, $c_n = -\frac{1}{2} \cdot c_{n-1}$ 15. 45 16. yes, a_{44} 17. 16 18. $5n$

19. $\frac{n+2}{3}$ 20. 12 21. 1 22. –5 23. $\mathbb{R}$ 24. use mathematical induction

EXERCISES 2

1. a. no b. yes c. yes 2. a. $2n+1$ b. $\sqrt{3}n + 1 - \sqrt{3}$ c. 0 d. $-\frac{3}{2}n - \frac{3}{2}$ e. $-n+1$ f. $7n + \sqrt{2} - 7$ g. $(b+3)n+b+4$

3. a. 2; $2n+1$ b. 2; $2n+2$ c. $\frac{5\sqrt{2}}{3}; \frac{5\sqrt{2}}{3}n - \frac{22\sqrt{2}}{3}$ d. –1; $-n$ e. 0; 8 f. $-\frac{20}{7}; -\frac{20}{7}n + \frac{162}{7}$ g. $\frac{1}{2}; \frac{1}{2}n - \frac{1}{2}$

h. $\frac{-x+3y}{6}; \frac{-x+3y}{6}n + \frac{14x-12y}{6}$ 4. a. $7n-9$ b. $4n-27$ 5. a. $-\frac{75}{4}, -\frac{23}{2}, -\frac{17}{4}; \frac{41}{4}, \frac{35}{2}, \frac{99}{4}$

b. 17, 21, 25, 29, 33, 37, 41 6. no 7. a. a_{16} b. a_8 8. a. $\mathbb{R} \setminus \{0, 1\}$ b. 6 c. 0 9. 12 10. 68 11. 5 12. a. 52

b. 3780 c. $\frac{11}{6}$ d. 7 e. –460 f. 80 g. 19 h. –210 i. 20 j. 125 k. –320 l. 19 m. $-\frac{67}{6}$ n. $-\frac{9}{2}$ o. 528 p. 235

13. yes 14. 162 15. 8775 16. 1, 9, 17 17. $4a$ 18. 49 19. $\frac{23}{4}$ 20. 110 21. 160 22. $\frac{77}{90}$ 23. $4n+1$ 24. 630

25. a. \$35 000 b. \$187 500 26. 2340 27. 470 28. no 29. a. 36.5 km b. 456.5 km 30. 8 hours 31. \$2737.50

32. $1+\frac{\sqrt{6}}{6}, 1, 1-\frac{\sqrt{6}}{6}$ or $1-\frac{\sqrt{6}}{6}, 1, 1+\frac{\sqrt{6}}{6}$ 33. use the arithmetic mean formula 34. 1 35. consider $n = 4, 5, 6$

36. 5000 37. 26 38. $\frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \ldots, \frac{25}{16}$ 39. n^2 40. let $m = n + k$ and consider the sum formula

41. consider the sum formula for each term 42. 263 700 43. 66 570 44. ±24 45. 80 200 46. $\frac{a+c}{2} \cdot n$

EXERCISES 3

1. a. yes b. no c. no 2. a. $5 \cdot 2^{n-1}$ b. $-3 \cdot 2^{1-n}$ c. 10^{4-n} d. $(\sqrt{3})^n$ e. 2^{n+1} f. 3^{2-n} or $3 \cdot (-3)^{1-n}$ g. 2^{-2n+11} h. 5 or $5 \cdot (-1)^{n-1}$ i. $(\sqrt{2})^{n+1}$ 3. a. –1 or 1 b. 324, 108; 12 or 324, –108; –12 4. 5^{n-2} 5. $-3, 3\sqrt{3}, -9, 9\sqrt{3}$ 6. –3 or 3 7. 12 8. $-\frac{1}{5}$ or $\frac{1}{5}$ 9. $\frac{8}{3}$ 10. $\frac{108}{7}$ 11. no 12. a. $-\frac{129}{2}$ b. 6141 c. $\frac{4681}{512}$ or $-\frac{3641}{512}$ d. 555 13. 6138 14. $\frac{3}{2}(\frac{5}{3})^n - \frac{3}{2}$ 15. $-\frac{1}{2}$ or $\frac{1}{2}$ 16. 5460 17. 780 18. 103.17 minutes 19. 8 20. express each number as the sum of the terms of a geometric sequence, e.g. $666 = 6 \cdot 10^2 + 6 \cdot 10 + 6$ 21. a. 6 b. $-\frac{2}{3}$ c. $\frac{4}{5}$ 22. a. 81 b. $\frac{9}{10}$ c. $\frac{49}{4}$ d. doesn't exist 23. a. $\frac{7}{33}$ b. $5\frac{47}{330}$ c. $-3\frac{202}{999}$ d. $2\frac{13}{198}$ 24. $\frac{1}{2}$ 25. 405, –270, 180, –120 26. a. $4\sqrt{2} + 8$ b. 2 27. 150 28. 12 29. 2; $\frac{1}{3}$ 30. $-\frac{1+\sqrt{5}}{2}$ 31. –3 32. $\frac{1}{(1-x)^2}$ 33. 0,0 or $\frac{10}{3}, \frac{4}{3}$ or $-\frac{3}{4}, -\frac{3}{10}$ 34. 5 or 20 35. –6, 6, 18 or 10, 6, 2 36. –2 37. 18, 12, 6, 3 or $\frac{9}{4}, \frac{27}{4}, \frac{45}{4}, \frac{75}{4}$ 38. write each factor in terms of b_1 and q 39. 40.5 40. 18 or 2 41. 2, 6, 18 or 18, 6, 2 42. solve arithmetic mean formula for b 43. 33 333 333

ANSWERS TO TESTS

TEST 1

1.	D	9.	B
2.	C	10.	D
3.	A	11.	C
4.	E	12.	E
5.	B	13.	E
6.	E	14.	C
7.	A	15.	E
8.	D	16.	B

TEST 2

1.	A	9.	E
2.	B	10.	D
3.	D	11.	A
4.	B	12.	D
5.	B	13.	C
6.	C	14.	C
7.	A	15.	B
8.	B	16.	B

TEST 3

1.	E	9.	B
2.	D	10.	D
3.	A	11.	B
4.	D	12.	B
5.	B	13.	C
6.	B	14.	E
7.	C	15.	C
8.	C	16.	B

GLOSSARY

A

area: any flat, curved, or irregular expanse of a surface.

average: also called arithmetic mean. The result obtained by adding the numbers or quantities in a set and dividing the total by the number of members in the set.

B

base: the number whose powers are expressed.

C

consecutive: following one another without interruption; successive.

criteria: a standard by which something can be judged or decided.

D

decimal: a fraction that has a denominator of a power of ten.

decrease: to make or become smaller in size.

denominator: the divisor of a fraction.

digit: each of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 which form a natural number.

E

e.g.: abbreviation for *exempli gratia* (in Latin, 'for example').

element: one of the objects or numbers that together constitute a set.

equilateral: having all sides of equal length.

etc.: abbreviation for *et cetera*.

even: a natural number which is divisible by two.

exclusive: excluding the limits specified.

existence: the fact or state of existing, being.

exponent: a number or variable placed as a superscript to the right of another number or quantity indicating the number of times the number or quantity is to be multiplied by itself.

F

Fibonacci: Leonardo Fibonacci, also called Leonardo of Pisa. A thirteenth-century Italian mathematician who popularized the decimal system in Europe.

Fibonacci sequence: the infinite sequence of numbers, 1, 1, 2, 3, 5, 8, etc., in which each number is the sum of the previous two. The sequence is named after Leonardo Fibonacci, who studied its properties.

finite: having a countable number of elements.

formula: a general relationship, principle, or rule stated, often as an equation, in the form of symbols.

fraction: a ratio of two expressions or numbers other than zero.

function: a relation between two sets that associates a unique element of the second with each element of the first.

I

i.e.: abbreviation for *id est* (in Latin, 'in other words').

inclusive: including the limits specified.

increase: to make or become greater in size.

infinite: having an unlimited or uncountable number of elements.

integer: any number which is a member of the set {...–3, –2, –1, 0, 1, 2, 3, ...}.

interval: a set containing all real numbers or points between two given numbers or points, called the endpoints. A closed interval includes the endpoints, but an open interval does not.

L

linear: of or relating to the first degree.

M

mean: another name for average.

monotone: consistently increasing or decreasing in value.

multiple: the product of a given number and any other one number.

N

natural number: any number which is a member of the set {1, 2, 3, ...}.

negative: less than zero.

notation: any series of signs or symbols used to represent quantities or elements in mathematics.

numerator: the dividend of a fraction.

O

odd: a natural number which is not divisible by two.

P

parabola: the graph of a quadratic function.

pentagon: a polygon with five sides.

perimeter: the curve or lines enclosing a plane area.

piecewise: a formula or function which is defined in pieces.

polygon: a closed plane figure bounded by three or more straight sides that meet in pairs at the vertices, and do not intersect other than at these vertices.

positive: more than zero.

prime number: a natural number that is only divisible by itself or 1, such as 2, 3, 7, and 11.

pyramid: a solid having a polygonal base and triangular sides that meet at a common vertex.

Pythagoras: a Greek philosopher and mathematician who lived in the sixth century B.C. He founded a religious brotherhood, which greatly influenced the development of mathematics and its application to music and astronomy.

Pythagorean: a follower of Pythagoras.

Q

quadratic: of or relating to the second power.

quadrilateral: a polygon which has four sides.

R

ratio: a quotient of two numbers or quantities.

real number: any rational or irrational number.

recursion: the application of a function to its own values to generate an infinite sequence of values.

repeating decimal: a rational number that contains a pattern of digits which repeats after the decimal point.

S

sequence: an ordered set of numbers in one-to-one correspondence with the natural numbers 1 to n.

series: the sum of a finite or infinite sequence of numbers or quantities.

set: a collection of numbers, objects, etc.

successive: following one another without interruption.

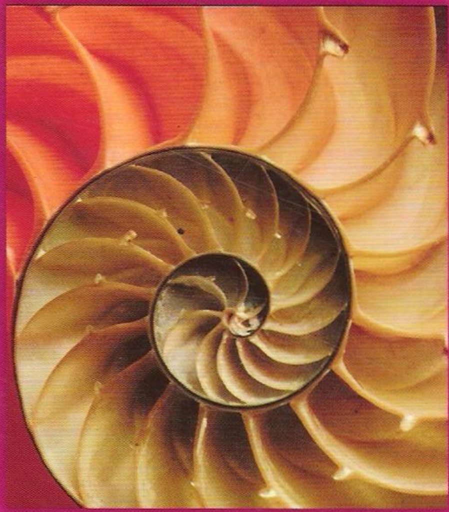
T

term: any of the separate elements of a sequence.

tetrahedron: a solid figure with four plane faces. A regular tetrahedron has faces that are equilateral triangles.

theorem: a statement or formula that can be deduced from the axioms of a formal system by means of its rules of inference.

trapezoid: a quadrilateral with only one pair of parallel sides.



Arithmetic and Geometric SEQUENCES

MODULAR SYSTEM

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PERIODIC

Name — Symbol

Iron Fe
55.85 26

Atomic mass — Atomic number

1	2		3	4	5	6	7	8	9
Hydrogen H 1.01 1									
Lithium Li 6.94 3	Beryllium Be 9.01 4								
Sodium Na 22.99 11	Magnesium Mg 24.31 12								
Potassium K 39.10 19	Calcium Ca 40.08 20		Scandium Sc 44.96 21	Titanium Ti 47.88 22	Vanadium V 50.94 23	Chromium Cr 52.00 24	Manganese Mn 54.94 25	Iron Fe 55.85 26	Cobalt Co 58.93 27
Rubidium Rb 85.47 37	Strontium Sr 87.62 38		Yttrium Y 88.91 39	Zirconium Zr 91.22 40	Niobium Nb 92.91 41	Molybdenum Mo 95.94 42	Technetium Tc (98) 43	Ruthenium Ru 101.07 44	Rhodium Rh 102.91 45
Cesium Cs 132.91 55	Barium Ba 137.33 56	Lanthanides * 57-70	Lutetium Lu 174.97 71	Hafnium Hf 178.49 72	Tantalum Ta 180.95 73	Tungsten W 183.84 74	Rhenium Re 186.21 75	Osmium Os 190.2 76	Iridium Ir 192.22 77
Francium Fr {223} 87	Radium Ra (226) 88	Actinides ** 89-102	Lawrencium Lr (262) 103	Rutherfordium Rf (261) 104	Dubnium Db (262) 105	Seaborgium Sg (266) 106	Bohrium Bh (264) 107	Hassium Hs (269) 108	Meitnerium Mt (268) 109

* Lanthanides	Lanthanum La 138.91 57	Cerium Ce 140.12 58	Praseodymium Pr 140.91 59	Neodymium Nd 144.24 60	Promethium Pm (145) 61	Samarium Sm 150.36 62
** Actinides	Actinium Ac (227.03) 89	Thorium Th 232.04 90	Protactinium Pa 231.04 91	Uranium U 238.03 92	Neptunium Np (237.05) 93	Plutonium Pu (244) 94

TABLE

10	11	12	13	14	15	16	17	18
								Helium He 4.0026 2
			Boron B 10.81 5	Carbon C 12.01 6	Nitrogen N 14.0067 7	Oxygen O 15.9994 8	Fluorine F 18.9984 9	Neon Ne 20.18 10
			Aluminum Al 26.98 13	Silicon Si 28.09 14	Phosphorus P 30.97 15	Sulfur S 32.07 16	Chlorine Cl 35.45 17	Argon Ar 39.95 18
Nickel Ni 58.69 28	Copper Cu 63.55 29	Zinc Zn 65.41 30	Gallium Ga 69.723 31	Germanium Ge 72.64 32	Arsenic As 74.92 33	Selenium Se 78.96 34	Bromine Br 79.90 35	Krypton Kr 83.80 36
Palladium Pd 106.42 46	Silver Ag 107.87 47	Cadmium Cd 112.41 48	Indium In 114.817 49	Tin Sn 118.71 50	Antimony Sb 121.76 51	Tellurium Te 127.60 52	Iodine I 126.90 53	Xenon Xe 131.29 54
Platinum Pt 195.08 78	Gold Au 196.97 79	Mercury Hg 200.59 80	Thallium Tl 204.383 81	Lead Pb 207.19 82	Bismuth Bi 208.980 83	Polonium Po (209) 84	Astatine At (210) 85	Radon Rn (222) 86
Darmstadium Uun (271) 110	Unununium Uuu (272) 111	Ununbium Uub (285) 112	113	Ununquadium Uuq 289 114	115	116	117	118

Europium Eu 151.96 63	Gadolinium Gd 157.25 64	Terbium Tb 158.92 65	Dysprosium Dy 162.50 66	Holmium Ho 164.93 67	Erbium Er 167.26 68	Thulium Tm 168.93 69	Ytterbium Yb 173.04 70
Americium Am (243) 95	Curium Cm (247) 96	Berkelium Bk (247) 97	Californium Cf (251) 98	Einsteinium Es (252) 99	Fermium Fm (257) 100	Mendelevium Md (258) 101	Nobelium No (259) 102